DYNAMICAL SYSTEMS DUAL TO INTERACTIONS AND GRAPH C^*-ALGEBRAS

B. K. KWAŚNIEWSKI

Abstract. For a class of Exel's interactions $(\mathcal{V},\mathcal{H})$ over a unital C^*-algebra A we define dual topological dynamical systems $(\widehat{\mathcal{V}},\widehat{\mathcal{H}})$ on the spectrum $\widehat{A}$ of A and use them to obtain uniqueness theorem, ideal lattice description and simplicity criteria for the associated crossed products $C^*(A,\mathcal{V},\mathcal{H})$. We show that the graph C^*-algebra $C^*(E)$ of a finite graph E is a natural example of such a crossed product on the core C^*-algebra $\mathcal{F}_E$. By describing the corresponding dual system $(\widehat{\mathcal{V}},\widehat{\mathcal{H}})$ on $\widehat{\mathcal{F}}_E$ we find new dynamical characterizations of Cuntz-Krieger uniqueness property, gauge-invariance of all ideals and simplicity of $C^*(E)$. We also characterize those n for which $(\mathcal{V}^n,\mathcal{H}^n)$ is again an interaction.

1. Introduction

In [10] R. Exel extended celebrated M. V. Pimsner's construction [23] of the so-called Cuntz-Pimsner algebras by introducing an intriguing and promising new concept of a generalized C^*-correspondence. The leading example in [10] arises from interactions. These are pairs $(\mathcal{V},\mathcal{H})$ of positive linear maps on a C^*-algebra A that are "symmetrized" generalizations of C^*-dynamical systems, i.e. pairs $(\alpha,\mathcal{L})$ consisting of an endomorphism α and its transfer operator $\mathcal{L}$, [9]. One can think of many examples of interactions naturally appearing in various problems, cf. [11], [13], [12]. However, not until the very recent paper [12], where it is shown that a C^*-algebra $\mathcal{O}_{n,m}$ (generalizing Cuntz algebra $\mathcal{O}_n$) is Morita equivalent to the crossed product $C^*(A,\mathcal{V},\mathcal{H})$ of an interaction $(\mathcal{V},\mathcal{H})$ on a commutative C^*-algebra A, there are no significant applications of $C^*(A,\mathcal{V},\mathcal{H})$ in the case $(\mathcal{V},\mathcal{H})$ is not a C^*-dynamical system.

The purpose of the present article is two fold. Firstly, we aim at describing the structure of $C^*(A,\mathcal{V},\mathcal{H})$ for an accessible class of interactions which might be a considerable step in understanding this new object. Secondly, we consider a problem where graph C^*-algebras $C^*(E)$ naturally arise as crossed products $C^*(A,\mathcal{V},\mathcal{H})$ of interactions which are non-trivial in the sense that both $\mathcal{V}$ and $\mathcal{H}$ are not multiplicative. Moreover, the interactions involved seem to be to a very extend canonical and perhaps there are reasons to consider them (in opposition to the usual canonical cp maps, cf. e.g. [16]) as alternative candidates for non-commutative one-sided Markov shifts on $C^*(E)$.

More specifically: many will agree that the Cuntz-Krieger uniqueness theorem is the central tool in graph C^*-algebras theory. It characterizes the graphs E for which every representation of $C^*(E)$ that faithfully represents E is faithful on $C^*(E)$. The gauge-invariant uniqueness theorems indicate that this property, called condition (L) in [20] (originally condition (I) in [8]), could or even should be expressed in terms of the associated gauge circle action γ on $C^*(E)$. Such a description would yield a very good candidate for a general condition responsible for the uniqueness property in C^*-algebras equipped with circle (or even more general) actions, cf. [19]. Inspired by this way of thinking the author proved in [17]

2010 *Mathematics Subject Classification.* 46L55.

Key words and phrases. interaction, graph algebra, topological freeness, crossed product, Hilbert bimodule.

This work was in part supported by Polish National Science Centre grant number DEC-2011/01/D/ST1/04112.

a uniqueness theorem for a crossed product $A \rtimes_X \mathbb{Z}$ [1] of a Hilbert bimodule (A, X) under the assumption that the induced representation functor X-Ind acts topologically freely on the spectrum $\widehat{A}$ of A. Since the gauge circle action γ on $C^*(E)$ is semi-saturated it follows from [1, Thm. 3.1] that $C^*(E) \cong B_0 \rtimes_{B_1} \mathbb{Z}$ where B_k stands for the k-th spectral subspace of γ and (B_0, B_1) is a Hilbert bimodule in a natural manner. Thus one faces the following question:

> What is the relationship between the Cuntz-Krieger uniqueness theorem for $C^*(E)$ and uniqueness theorem for $B_0 \rtimes_{B_1} \mathbb{Z}$?

In order to answer this question thoroughly one needs to describe the dynamics of B_1-Ind on the spectrum of the core C^*-algebra $\mathcal{F}_E = B_0$, and we do this in detail under the assumption that E is finite (this in particular allows us to stay in the category of unital C^*-algebras but probably could be omitted). The idea is to use a certain partial isometry $s \in C^*(E)$ which in a more or less implicit way appears for instance in [9], [2], [7], [15], and for which we have $B_1 = \mathcal{F}_E s \mathcal{F}_E$. In particular, if E has no sources, s is an isometry, $\alpha(\cdot) = s(\cdot)s^*$ is a monomorphism with a hereditary range, and $C^*(E)$ can be naturally considered as a crossed product of $\mathcal{F}_E$ by α, see [2], [15]. However, if we allow sources the both mappings

$$\mathcal{V}(\cdot) = s(\cdot)s^*, \qquad \mathcal{H}(\cdot) = s^*(\cdot)s \tag{1}$$

in general are not multiplicative. Still $(\mathcal{V}, \mathcal{H})$ forms an interaction over $\mathcal{F}_E$ and one of our goals is to show how the above mentioned problem can be solved by describing a topological dynamical system $(\widehat{\mathcal{V}}, \widehat{\mathcal{H}})$ dual to $(\mathcal{V}, \mathcal{H})$.

We start in section 2 by proving that for a class of interactions $(\mathcal{V}, \mathcal{H})$ with hereditary ranges (we call them complete interactions), the crossed product $C^*(A, \mathcal{V}, \mathcal{H})$ can be viewed as a crossed product by a certain Hilbert bimodule X. We show that the partial homeomorphism X-Ind introduced in [17] coincides with a partial map $\widehat{\mathcal{H}} = \widehat{\mathcal{V}}^{-1}$ on $\widehat{A}$ where $\widehat{\mathcal{V}}$ and $\widehat{\mathcal{H}}$ are naturally defined duals to $\mathcal{V}$ and $\mathcal{H}$. Thus applying general theorems of [17] we obtain uniqueness theorem, ideal lattice description (under the assumption $\widehat{\mathcal{V}}$ is free) and simplicity criteria for $C^*(A, \mathcal{V}, \mathcal{H})$. These results are interesting in their own rights and in particular they apply to complete dynamical systems $(\alpha, \mathcal{L})$ which model many C^*-algebraic constructions, see [2].

In section 3 we associate to each finite graph E a partial isometry $s \in C^*(E)$ such that for the interaction given by (1) we have $C^*(E) \cong C^*(\mathcal{F}_E, \mathcal{V}, \mathcal{H})$. We prove that paths in E give rise to a dense subset of representation in $\widehat{\mathcal{F}_E}$, and $\widehat{\mathcal{V}}$ acts on these representations as a quotient of the one-sided Markov shift. This allows us to explain how the existence of loops in E effects the dynamics of $\widehat{\mathcal{V}}$. In particular, $\widehat{\mathcal{V}}$ is topologically free if and only if E satisfies (L) and we characterize in terms of $\widehat{\mathcal{V}}$ the so called condition (K) [21], [4] and simplicity criteria for $C^*(E)$ using the results for the crossed product of interactions. Additionally, in subsection 3.3, we determine the values n for which $(\mathcal{V}^n, \mathcal{H}^n)$ is an interaction. This allows one to construct interactions with (almost) arbitrary distribution of such values of n and shows that generic $(\mathcal{V}, \mathcal{H})$, as in [12], is not a part of a group interaction [11].

1.1. **Background and notation.**

Throughout A is a C^*-algebra which (starting from section 2) will always be unital. By homomorphisms, epimorphisms, etc. between C^*-algebras we always mean *-preserving maps. All ideals in C^*-algebras are assumed to be closed and two sided. We adhere to the convection that $\beta(A, B) = \overline{\operatorname{span}}\{\beta(a, b) \in C : a \in A, b \in B\}$ for maps $\beta\colon A \times B \to C$ such as inner products, multiplications or representations.

As in [17] we say that a partial homeomorphism φ of a topological space M, i.e. a homeomorphism whose domain Δ and image $\varphi(\Delta)$ are open subsets of M, is *topologically free* if for any $n > 0$ the set of fixed points for φ^n (on its natural domain) has empty interior. A set V is *φ-invariant* if $\varphi(V \cap \Delta) = V \cap \varphi(\Delta)$. If there are no non-trivial closed invariant

sets, then φ is called *minimal*, and φ is said to be *free*, if it is topologically free on every closed invariant set (in the Hausdorff space case this amounts to requiring that φ has no periodic points).

Following [6, 1.8] and [1] by a *Hilbert bimodule over* A we mean X which is both a left Hilbert A-module and a right Hilbert A-module with respective inner products $\langle\cdot,\cdot\rangle_A$ and ${}_A\langle\cdot,\cdot\rangle$ satisfying the so-called *imprimitivity condition*: $x\cdot\langle y,z\rangle_A = {}_A\langle x,y\rangle\cdot z$, for all $x,y,z\in X$. A *covariant representation* of X is a pair (π_A,π_X) consisting of a homomorphism $\pi_A : A\to\mathcal{B}(H)$ and a linear map $\pi_X : X\to\mathcal{B}(H)$ such that

$$\pi_X(ax)=\pi_A(a)\pi_X(x), \qquad \pi_X(xa)=\pi_X(x)\pi_A(a), \tag{2}$$

$$\pi_A(\langle x,y\rangle_A)=\pi_X(x)^*\pi_X(y), \qquad \pi_A({}_A\langle x,y\rangle)=\pi_X(x)\pi_X(y)^*, \tag{3}$$

for all $a\in A$, $x,y\in X$. The *crossed product* $A\rtimes_X\mathbb{Z}$ is a C^*-algebra generated by a copy of A and X universal with respect to covariant representations of X, see [1]. It is equipped with the *circle gauge action* $\gamma=\{\gamma_z\}_{z\in\mathbb{T}}$ given on generators by $\gamma_z(a)=a$ and $\gamma_z(x)=zx$, for $a\in A$, $x\in X$, $z\in\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$.

We will abuse the language (in the standard manner) and denote by π both an irreducible representation of A and its equivalence class in the spectrum $\widehat{A}$ of A. This should not cause confusion. In particular, for a Hilbert bimodule X the functor X-Ind preserves such classes, cf. e.g. [24]. We recall that X-Ind maps a representation $\pi : A\to\mathcal{B}(H)$ to a representation $X\text{-Ind}(\pi) : A\to\mathcal{B}(X\otimes_\pi H)$ where the Hilbert space $X\otimes_\pi H$ is generated by simple tensors $x\otimes_\pi h$, $x\in X$, $h\in H$, satisfying $\langle x_1\otimes_\pi h_1, x_2\otimes_\pi h_2\rangle=\langle h_1,\pi(\langle x_1,x_2\rangle_A)h_2\rangle$, and

$$X\text{-Ind}(\pi)(b)(x\otimes_\pi h)=(bx)\otimes_\pi h.$$

Identifying the spectra of the ideals $\langle X,X\rangle_A$ and ${}_A\langle X,X\rangle$ in A with open subsets in $\widehat{A}$ we may treat $X\text{-Ind} : \widehat{\langle X,X\rangle_A}\to\widehat{{}_A\langle X,X\rangle}$ as a partial homeomorphism of $\widehat{A}$, see [17]. The results of [17] can be summarized as follows.

Theorem 1.1. *Let* $\widehat{h}:=X\text{-Ind}$ *be as described above.*

i) *If* $\widehat{h}$ *is topologically free, then for every faithful covariant representation* (π_A,π_X) *of* X *integrates to the faithful representation of* $A\rtimes_X\mathbb{Z}$.
ii) *If* $\widehat{h}$ *is free, then* $J\mapsto\widehat{J\cap A}$ *is a lattice isomorphism between ideals in* $A\rtimes_X\mathbb{Z}$ *and open invariant sets in* $\widehat{A}$.
iii) *If* $\widehat{h}$ *is topologically free and minimal, then* $A\rtimes_X\mathbb{Z}$ *is simple.*

Remark 1.2. The map $\widehat{h}$ is a lift of the partial homeomorphism $h:\operatorname{Prim}\langle X,X\rangle_A\to\operatorname{Prim}{}_A\langle X,X\rangle$ of $\operatorname{Prim}A$ where h is the restriction of the Rieffel isomorphism between the ideal lattices of $\langle X,X\rangle_A$ and ${}_A\langle X,X\rangle$, cf. [17, Rem. 2.3], [24]. Plainly, topological freeness of $(\operatorname{Prim}(A),h)$ implies the topological freeness of $(\widehat{A},\widehat{h})$, but the converse is not true and we give a very good example of this phenomena based on Cuntz algebras $\mathcal{O}_n$, see Example 3.14 below.

The Hilbert bimodule X may be treated a as a generalized C^*-correspondence described in terms of [10, Prop 7.6] by the triple (X,λ,ρ) where we consider X as a ${}_A\langle X,X\rangle$-$\langle X,X\rangle_A$-Hilbert bimodule and define homomorphisms $\lambda : A\to{}_A\langle X,X\rangle$ and $\rho : A\to\langle X,X\rangle_A$ to be (necessarily unique) extensions of the identity maps.

Proposition 1.3. *The crossed product* $A\rtimes_X\mathbb{Z}$ *of a Hilbert bimodule* X *is naturally isomorphic to the covariance algebra* $C^*(A,X)$*, defined in* [10, 7.12]*, for* X *treated as a generalized correspondence.*

Proof. The Toeplitz algebra $\mathcal{T}(A,X)$ defined in [10, 7.7] is a universal C^*-algebra generated by a copy of A and X subject to all A-A-bimodule relations plus the relation

$$xy^*z=x\langle y,z\rangle_A={}_A\langle x,y\rangle z, \qquad x,y,z\in X. \tag{4}$$

The C^*-algebra $C^*(A,X)$ is the quotient $\mathcal{T}(A,X)/J$ where $J = J_\ell + J_r$ and J_ℓ (respectively J_r) is an ideal in $\mathcal{T}(A,X)$ generated by the elements $a-k$ such that $a \in (\ker\lambda)^\perp$, $k \in XX^*$ (resp. $a \in (\ker\rho)^\perp$, $k \in X^*X$) and

$$ax = kx \quad \text{(or resp. } xa = xk) \quad \text{for all } x \in X. \tag{5}$$

Note that $(\ker\lambda)^\perp = {}_A\langle X, X\rangle$ and $(\ker\rho)^\perp = \langle X, X\rangle_A$. By (4), XX^* and X^*X are C^*-subalgebras of $\mathcal{T}(A,X)$ and hence relations (5) determine k uniquely (when a is fixed). It follows that

$$J_\ell = \overline{\operatorname{span}}\{{}_A\langle x,y\rangle - xy^* : x,y \in X\}, \qquad J_r = \overline{\operatorname{span}}\{\langle x,y\rangle_A - x^*y : x,y \in X\},$$

because if (for instance) $a-k \in J_\ell$ where $a = \sum_{i=1}^n {}_A\langle x_i, y_i\rangle \in (\ker\lambda)^\perp$ and $k \in X^*X$, then by (4), $ax = \sum_{i=1}^n x_i y_i^* x$ for all $x \in X$ and thus $k = \sum_{i=1}^n x_i y_i^*$.

Accordingly $C^*(A,X)$ is a universal C^*-algebra generated by a homomorphic image of A and X subject to the same relations as the Hilbert bimodule X. □

2. Complete interactions and their crossed products

2.1. **Interactions and C^*-dynamical systems.** It is instructive to consider interactions as generalization of pairs $(\alpha, \mathcal{L})$ consisting of an endomorphism $\alpha : A \to A$ and its *transfer operator* [9], i.e. positive linear map $\mathcal{L} : A \to A$ such that $\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b)$, $a, b \in A$ (then $\mathcal{L}$ is automatically continuous, *-preserving and by passing to adjoints one also gets $\mathcal{L}(b\alpha(a)) = \mathcal{L}(b)a$, $a, b \in A$). A transfer operator $\mathcal{L}$ is said to be *non-degenerate* if $\alpha(\mathcal{L}(1)) = \alpha(1)$, or equivalently [9, Prop. 2.3], if $\mathcal{E}(a) := \alpha(\mathcal{L}(a))$ is a conditional expectation from A onto $\alpha(A)$. It is important, see [18], that the range of a non-degenerate transfer operator $\mathcal{L}$ coincides with the annihilator $(\ker\alpha)^\perp$ of the kernel of α and $\mathcal{L}(1)$ is unit in $\mathcal{L}(A) = (\ker\alpha)^\perp$.

Definition 2.1. A pair $(\alpha, \mathcal{L})$ where $\mathcal{L} : A \to A$ is a non-degenerate transfer operator for an endomorphism $\alpha : A \to A$ will be called a *C^*-dynamical system.*

A dissatisfaction concerning asymmetry in $(\alpha, \mathcal{L})$ (α is multiplicative while $\mathcal{L}$ is "merely" positive linear) lead the author of [10] to the following

Definition 2.2 ([10], Defn. 3.1)**.** The pair $(\mathcal{V}, \mathcal{H})$ of positive linear maps $\mathcal{V}, \mathcal{H} : A \to A$ is called an *interaction* over A if

(i) $\mathcal{V} \circ \mathcal{H} \circ \mathcal{V} = \mathcal{V}$,
(ii) $\mathcal{H} \circ \mathcal{V} \circ \mathcal{H} = \mathcal{H}$,
(iii) $\mathcal{V}(ab) = \mathcal{V}(a)\mathcal{V}(b)$, if either a or b belong to $\mathcal{H}(A)$,
(iv) $\mathcal{H}(ab) = \mathcal{H}(a)\mathcal{H}(b)$, if either a or b belong to $\mathcal{V}(A)$.

Remark 2.3. In general an interaction $(\mathcal{V}, \mathcal{H})$ (or even a C^*-dynamical system $(\alpha, \mathcal{L})$) does not yield a semigroup of interactions, and all the more a group interaction in the sense of [11]. Namely, the powers $(\mathcal{V}^n, \mathcal{H}^n)$ may not be interactions, and this will be a generic case in our example arising from graph C^*-algebras, cf. Proposition 3.6 below.

Let $(\mathcal{V}, \mathcal{H})$ be an interaction. By [10, Prop. 2.6, 2.7], $\mathcal{V}(A)$ and $\mathcal{H}(A)$ are C^*-subalgebras of A, $\mathcal{E}_\mathcal{V} := \mathcal{V} \circ \mathcal{H}$ is a conditional expectation onto $\mathcal{V}(A)$, $\mathcal{E}_\mathcal{H} := \mathcal{H} \circ \mathcal{V}$ is a conditional expectation onto $\mathcal{H}(A)$, and the mappings

$$\mathcal{V} : \mathcal{H}(A) \to \mathcal{V}(A), \qquad \mathcal{H} : \mathcal{V}(A) \to \mathcal{H}(A)$$

are isomorphisms, each being the inverse of the other. Actually we have

Proposition 2.4. *The relations $\mathcal{E}_\mathcal{V} = \mathcal{V} \circ \mathcal{H}$, $\mathcal{E}_\mathcal{H} = \mathcal{H} \circ \mathcal{V}$, $\theta = \mathcal{V}|_{\mathcal{E}_\mathcal{H}(A)}$ yield a one-to-one correspondence between interactions $(\mathcal{V}, \mathcal{H})$ and triples $(\theta, \mathcal{E}_\mathcal{V}, \mathcal{E}_\mathcal{H})$ consisting of two conditional expectations $\mathcal{E}_\mathcal{V}, \mathcal{E}_\mathcal{H}$ and an isomorphism $\theta : \mathcal{E}_\mathcal{H}(A) \to \mathcal{E}_\mathcal{V}(A)$.*

Proof. It suffices to verify that for the triple $(\theta, \mathcal{E}_\mathcal{V}, \mathcal{E}_\mathcal{H})$ as in the assertion $\mathcal{V}(a) := \theta(\mathcal{E}_\mathcal{H}(a))$ and $\mathcal{H}(a) := \theta^{-1}(\mathcal{E}_\mathcal{V}(a))$ form an interaction and this is straightforward. □

The algebras involved in an interaction are unital.

Lemma 2.5. *For any interaction $(\mathcal{V},\mathcal{H})$ the elements $\mathcal{V}(1)$ and $\mathcal{H}(1)$ are units in $\mathcal{V}(A)$ and $\mathcal{H}(A)$, respectively (in particular, they are projections).*

Proof. Let us observe that

$$\begin{aligned}\mathcal{E}_{\mathcal{V}}(1) &= \mathcal{V}(\mathcal{H}(1)) = \mathcal{V}(\mathcal{H}(1)1) = \mathcal{V}(\mathcal{H}(1))\mathcal{V}(1) = \mathcal{V}(\mathcal{H}(1))\mathcal{V}\big(\mathcal{H}(\mathcal{V}(1))\big)\\ &= \mathcal{V}\big(\mathcal{H}(1)\mathcal{H}(\mathcal{V}(1))\big) = \mathcal{V}\big(\mathcal{H}(1\mathcal{V}(1))\big) = \mathcal{V}\big(\mathcal{H}(\mathcal{V}(1))\big) = \mathcal{V}(1)\end{aligned}$$

and thus we have $\mathcal{V}(a) = \mathcal{E}_{\mathcal{V}}(\mathcal{V}(a)) = \mathcal{E}_{\mathcal{V}}(1\mathcal{V}(a)) = \mathcal{E}_{\mathcal{V}}(1)\mathcal{V}(a) = \mathcal{V}(1)\mathcal{V}(a)$. Hence $\mathcal{V}(1)$ is the unit in $\mathcal{V}(A)$ and the similar argument works for $\mathcal{H}$. □

The following generalizes [10, Prop. 3.4].

Proposition 2.6. *Any C^*-dynamical system $(\alpha,\mathcal{L})$ is an interaction.*

Proof. Consider the conditions (i)-(iv) in Definition 2.2. Since $\alpha\circ\mathcal{L}\circ\alpha = \mathcal{E}\circ\alpha = \alpha$, (i) holds. For (ii) note that $\mathcal{L}(\alpha(\mathcal{L}(a))) = \mathcal{L}(1\alpha(\mathcal{L}(a))) = \mathcal{L}(1)\mathcal{L}(a) = \mathcal{L}(a)$ ($\mathcal{L}(1)$ is the unit in $\mathcal{L}(A)$, see [18, Prop. 1.5]). Condition (iii) is trivial, and (iv) holds because

$$\mathcal{L}(a\alpha(b)) = \mathcal{L}(a)b = \mathcal{L}(a)\mathcal{L}(1)b = \mathcal{L}(a)\mathcal{L}(1\alpha(b)) = \mathcal{L}(a)\mathcal{L}(\alpha(b)),$$

and by passing to adjoints we also get $\mathcal{L}(\alpha(b)a) = \mathcal{L}(\alpha(b))\mathcal{L}(a)$. □

As shown in [2] there is a very natural crossed product associated to the C^*-dynamical system $(\alpha,\mathcal{L})$ in the case the conditional expectation $\mathcal{E} = \alpha\circ\mathcal{L}$ is given by

$$\mathcal{E}(a) = \alpha(1)a\alpha(1), \qquad a \in A. \tag{6}$$

This crossed product coincides with the one introduced in [9] and is sufficient to cover many classic constructions, see [2]. A transfer operator satisfying (6) is called *complete* [3], [2]. By [18] a given endomorphism α admits a complete transfer operator $\mathcal{L}$ if and only if $\ker\alpha$ is a complementary ideal and $\alpha(A)$ is a hereditary subalgebra in A. In this case $\mathcal{L}$ is a unique non-degenerate transfer operator for α, cf. also [3], [2]. We naturally generalize the aforementioned concepts to interactions.

Definition 2.7. An interaction $(\mathcal{V},\mathcal{H})$ such that $\mathcal{V}(A)$ and $\mathcal{H}(A)$ are hereditary subalgebras of A will be called a *complete interaction.*

Proposition 2.8. *An interaction $(\mathcal{V},\mathcal{H})$ is complete if and only if $\mathcal{V}(A) = \mathcal{V}(1)A\mathcal{V}(1)$ and $\mathcal{H}(A) = \mathcal{H}(1)A\mathcal{H}(1)$ are corners in A. Moreover, for a complete interaction $(\mathcal{V},\mathcal{H})$ the following conditions are equivalent*

- i) *$(\mathcal{V},\mathcal{H})$ is a C^*-dynamical system (with a complete transfer operator),*
- ii) *$\mathcal{V}$ is multiplicative,*
- iii) $\ker\mathcal{V}$ *is an ideal in A,*
- iv) *$\mathcal{H}(A)$ is an ideal in A,*
- v) *$\mathcal{H}(1)$ lies in the center of A.*

Proof. For the first part of assertion apply Lemma 2.5 and notice that if B is a hereditary subalgebra of A and P is unit in B, then $B = PAP$. Suppose then that $(\mathcal{V},\mathcal{H})$ is a complete interaction. The implications i) $\Rightarrow$ ii) $\Rightarrow$ iii) and the equivalence iv) $\Leftrightarrow$ v) are clear.

iii) $\Rightarrow$ v). Note that (for any complete interaction) $(1-\mathcal{H}(1))A + A(1-\mathcal{H}(1)) \subset \ker\mathcal{V}$ and as $\mathcal{V}$ is isometric on $\mathcal{H}(1)A\mathcal{H}(1)$ we actually have $\ker\mathcal{V} = (1-\mathcal{H}(1))A + A(1-\mathcal{H}(1))$. Hence for any $a\in A$ we have $\mathcal{H}(1)a(1-\mathcal{H}(1)) \in \ker\mathcal{V}$ and if $\ker\mathcal{V}$ is an ideal, then $\mathcal{H}(1)a\big(1-\mathcal{H}(1)\big)a^*\mathcal{H}(1) \in (\ker\mathcal{V})\cap\mathcal{H}(1)A\mathcal{H}(1) = \{0\}$. Hence $\mathcal{H}(1)a(1-\mathcal{H}(1)) = 0$ which means that $\mathcal{H}(1)a = a\mathcal{H}(1)$.

v) $\Rightarrow$ i). By the first part of the assertion $\mathcal{E}_{\mathcal{H}}(a) = \mathcal{H}(1)a\mathcal{H}(1)$. Thus

$$\begin{aligned}\mathcal{V}(ab) &= \mathcal{V}(\mathcal{E}_{\mathcal{H}}(ab)) = \mathcal{V}(\mathcal{H}(1)ab\mathcal{H}(1)) = \mathcal{V}(a\mathcal{H}(1)b\mathcal{H}(1))\\ &= \mathcal{V}(a\mathcal{E}_{\mathcal{H}}(b)) = \mathcal{V}(a)\mathcal{V}(\mathcal{E}_{\mathcal{H}}(b)) = \mathcal{V}(a)\mathcal{V}(b).\end{aligned}$$

Hence $\mathcal{V}$ is an endomorphism of A, and $\mathcal{H}$ is its transfer operator because

$$\mathcal{H}(a\mathcal{V}(b)) = \mathcal{H}(a)\mathcal{H}(\mathcal{V}(b))) = \mathcal{H}(a)\mathcal{H}(1)b\mathcal{H}(1) = \mathcal{H}(a)b.$$

□

As in the case of C^*-dynamical systems each mapping in a complete interaction determines uniquely the other.

Proposition 2.9. *A positive linear map $\mathcal{V} : A \to A$ is a part of a non-zero complete interaction $(\mathcal{V}, \mathcal{H})$ if and only if $\|\mathcal{V}(1)\| = 1$, $\mathcal{V}(A)$ is a hereditary subalgebra of A and there is a projection $P \in A$ such that $\mathcal{V} : PAP \to \mathcal{V}(A)$ is an isomorphism.*

If this the case, then P and $\mathcal{H}$ are uniquely determined by $\mathcal{V}$ and we have

$$\mathcal{H}(a) := \mathcal{V}^{-1}(\mathcal{V}(1)a\mathcal{V}(1)) \tag{7}$$

where $\mathcal{V}^{-1}$ is the inverse to $\mathcal{V} : PAP \to \mathcal{V}(A)$.

Proof. The necessity follows from Proposition 2.8 and Lemma 2.5. For the sufficiency note that $\mathcal{V}(P)$ is a unit in $\mathcal{V}(A)$ and therefore $\mathcal{V}(A) = \mathcal{V}(P)A\mathcal{V}(P)$ (by hereditariness). In particular, $\mathcal{E}_{\mathcal{V}}(a) := \mathcal{V}(P)a\mathcal{V}(P)$ is a conditional expectation onto $\mathcal{V}(A)$. We put $\mathcal{E}_{\mathcal{H}}(a) := \mathcal{V}^{-1}(\mathcal{V}(a))$ where $\mathcal{V}^{-1}$ is the inverse to $\mathcal{V} : PAP \to \mathcal{V}(A)$. Then $\mathcal{E}_{\mathcal{H}}$ is an idempotent map of norm one because $\|\mathcal{E}_{\mathcal{H}}\| = \|\mathcal{V}\| = \|\mathcal{V}(1)\| = 1$. Hence $\mathcal{E}_{\mathcal{H}}$ is a conditional expectation onto PAP. The triple $(\mathcal{V}, \mathcal{E}_{\mathcal{V}}, \mathcal{E}_{\mathcal{H}})$ correspond, in the sense of Proposition 2.4, to a necessarily complete interaction $(\mathcal{V}, \mathcal{H})$ where $\mathcal{H}(a) = \mathcal{V}^{-1}(\mathcal{V}(P)a\mathcal{V}(P))$. In particular, it follows that $\mathcal{V}(P) = \mathcal{V}(1)$ by Lemma 2.5.

It remains to show the uniqueness of P. Suppose then that $(\mathcal{V}, \mathcal{H}_i)$, $i = 1, 2$, are complete interactions. For the projections $P_1 = \mathcal{H}_1(1)$ and $P_2 = \mathcal{H}_2(1)$ we have $\mathcal{V}(P_1P_2P_1) = \mathcal{V}(P_2) = \mathcal{V}(1) = \mathcal{V}(P_1) = \mathcal{V}(P_2P_1P_2)$. As $\mathcal{V}$ is injective on $\mathcal{H}_i(A) = P_iAP_i$, $i = 1, 2$, it follows that $P_1P_2P_1 = P_1$ and $P_2 = P_2P_1P_2$, which implies $P_1 = P_2$. □

2.2. **Crossed product of complete interactions.** We fix a complete interaction $(\mathcal{V}, \mathcal{H})$. In contrast to [10] we introduce the crossed product for $(\mathcal{V}, \mathcal{H})$, in a simpler way, without a use of the generalized C^*-correspondence constructed in [10].

Definition 2.10. A *covariant representation* of $(\mathcal{V}, \mathcal{H})$ is a pair (π, S) consisting of a non-degenerate representation $\pi : A \to \mathcal{B}(H)$ and an operator $S \in \mathcal{B}(H)$ such that

$$S\pi(a)S^* = \pi(\mathcal{V}(a)) \text{ and } S^*\pi(a)S = \pi(\mathcal{H}(a)) \text{ for all } a \in A$$

(S is necessarily a partial isometry by Lemma 2.5). The *crossed product of the interaction* $(\mathcal{V}, \mathcal{H})$ is the C^*-algebra $C^*(A, \mathcal{V}, \mathcal{H})$ generated by $i_A(A)$ and s where (i_A, s) is a universal covariant representation of $(\mathcal{V}, \mathcal{H})$. It is equipped with the *circle gauge action* determined by $\gamma_z(i_A(a)) = i_A(a)$, $a \in A$, and $\gamma_z(s) = zs$.

The above definition in an obvious way generalizes the crossed product for complete C^*-dynamical systems [2]. To see its coincidence with the one introduced in [10] we associate a Hilbert bimodule to $(\mathcal{V}, \mathcal{H})$ by adopting to our setting Exel's construction of his generalized C^*-correspondence.

We fix a complete interaction $(\mathcal{V}, \mathcal{H})$. Let $X_0 = A \odot A$ be the algebraic tensor product over the complexes, and let $\langle \cdot, \cdot \rangle_A$ and ${}_A\langle \cdot, \cdot \rangle$ be the A-valued sesqui-linear functions defined on $X_0 \times X_0$ by

$$\langle a \odot b, c \odot d \rangle_A = b^*\mathcal{H}(a^*c)d, \qquad {}_A\langle a \odot b, c \odot d \rangle = a\mathcal{V}(bd^*)c^*.$$

We consider the linear space X_0 as an A-A-bimodule with the natural module operations: $a \cdot (b \odot c) = ab \odot c$, $(a \odot b) \cdot c = a \odot bc$.

Proposition 2.11. *A quotient of X_0 becomes naturally a pre-Hilbert A-A-bimodule. More precisely*

i) *the space X_0 with a function $\langle \cdot, \cdot \rangle_A$ (respectively ${}_A\langle \cdot, \cdot \rangle$) becomes a right (respectively left) semi-inner product A-module.*

ii) *the corresponding semi-norms*
$$\|x\|_A := \|\langle x, x\rangle_A\|^{\frac{1}{2}} \quad \text{and} \quad {}_A\|x\| := \|{}_A\langle x, x\rangle\|^{\frac{1}{2}}$$
coincide on X_0 and thus the quotient space $X_0/\|\cdot\|$ obtained by modding out the vectors of length zero with respect to the seminorm $\|x\| := \|x\|_A = {}_A\|x\|$ is both a left and a right pre-Hilbert module over A.

iii) *denoting by $a \otimes b$ the canonical image of $a \odot b$ in the quotient space $X_0/\|\cdot\|$ we have*
$$ac \otimes b = a \otimes \mathcal{H}(c)b, \quad \text{if } c \in \mathcal{V}(A), \qquad a \otimes cb = a\mathcal{V}(c) \otimes b, \quad \text{if } c \in \mathcal{H}(A),$$
and $a \otimes b = a\mathcal{V}(1) \otimes \mathcal{H}(1)b$ for all $a, b \in A$.

iv) *the inner-products in $X_0/\|\cdot\|$ satisfy the imprimitivity condition.*

Proof. i) All axioms of A-valued semi-inner products for $\langle \cdot, \cdot \rangle_A$ and ${}_A\langle \cdot, \cdot \rangle$ except the non-negativity are straightforward, and to show the latter one may rewrite the proof of [10, Prop. 5.2] (just erase the symbol $e_{\mathcal{H}}$ or put $e_{\mathcal{H}} = \mathcal{H}(1)$).
ii) Similarly, the proof of [10, Prop. 5.4] implies that for $x = \sum_{i=1}^{n} a_i^* \odot b_i$, $a_i, b_i \in A$, we have

$$\|x\|_A = \|\mathcal{H}(aa^*)^{\frac{1}{2}}\mathcal{H}(\mathcal{V}(bb^*))^{\frac{1}{2}}\| = \|\mathcal{V}(\mathcal{H}(aa^*))^{\frac{1}{2}}\mathcal{V}(bb^*)^{\frac{1}{2}}\| = {}_A\|x\| \tag{8}$$

where $a = (a_1, ..., a_n)^T$ and $b = (b_1, ..., b_n)^T$ are viewed as column matrices.
iii) For the first part consult the proof of [10, Prop. 5.6]. The second part could be proved analogously. Namely, for every x, $y \in A$ we have
$$\langle x \otimes y, a \otimes b\rangle_A = y^*\mathcal{H}(x^*a)b = y^*\mathcal{H}(x^*a\mathcal{V}(1))\mathcal{H}(1)b = \langle x \otimes y, a\mathcal{V}(1) \otimes \mathcal{H}(1)b\rangle_A$$
which imply that $\|a \otimes b - a\mathcal{V}(1) \otimes \mathcal{H}(1)b\| = 0$.
iv) The form of imprimitivity condition allows one to restrict to the case of simple tensors. Using iii) we have
$$\begin{aligned} a \otimes b\langle c \otimes d, e \otimes f\rangle_A &= a \otimes bd^*\mathcal{H}(c^*e)f = a \otimes \mathcal{H}(1)bd^*\mathcal{H}(c^*e)f \\ &= a\mathcal{V}\Big(\mathcal{H}(1)bd^*\mathcal{H}(c^*e)\Big) \otimes f = a\mathcal{V}(\mathcal{H}(1)bd^*)\mathcal{V}(\mathcal{H}(c^*e)) \otimes f \\ &= a\mathcal{V}(bd^*)\mathcal{V}(1)c^*e\mathcal{V}(1) \otimes f = a\mathcal{V}(bd^*)c^*e \otimes f \\ &= {}_A\langle a \otimes b, c \otimes d\rangle e \otimes f. \end{aligned}$$
□

Definition 2.12. We call the completion X of the pre-Hilbert bimodule X_0 described in Proposition 2.11 a *Hilbert bimodule associated to* $(\mathcal{V}, \mathcal{H})$.

Remark 2.13. The Hilbert bimodule X could be obtained directly from the imprimitivity $\mathcal{K}_{\mathcal{V}}$-$\mathcal{K}_{\mathcal{H}}$-bimodule $\mathfrak{X}$ constructed by Exel in [10, Sec. 5] in the following way. By (8), X and $\mathfrak{X}$ coincide as Banach spaces, and since
$$\langle X, X\rangle_A = A\mathcal{H}(1)A, \qquad {}_A\langle X, X\rangle = A\mathcal{V}(1)A,$$
X could be consider as an imprimitivity $A\mathcal{V}(1)A$-$A\mathcal{H}(1)A$-bimodule. Furthermore, the mappings $\lambda_{\mathcal{V}} : A \to \mathcal{K}_{\mathcal{V}}$, $\lambda_{\mathcal{H}} : A \to \mathcal{K}_{\mathcal{V}}$, the author of [10] used to define an A-A-bimodule structure on $\mathfrak{X}$, when restricted respectively to $A\mathcal{V}(1)A$ and $A\mathcal{H}(1)A$ are isomorphisms. Hence we may use them to assume the identifications $\mathcal{K}_{\mathcal{V}} = A\mathcal{V}(1)A$ and $\mathcal{K}_{\mathcal{H}} = A\mathcal{H}(1)A$ and then the Exel's generalized correspondence and the Hilbert bimodule X coincide.

Now we are ready to identify the structure of $C^*(A,\mathcal{V},\mathcal{H})$.

Proposition 2.14. *We have a one-to-one correspondence between the covariant representations (π,S) of the interaction $(\mathcal{V},\mathcal{H})$ and covariant representations (π,π_X) of the Hilbert bimodule X associated to $(\mathcal{V},\mathcal{H})$. It is given by relations*

$$\pi_X(a\otimes b)=\pi(a)S\pi(b),\ x\in X,\qquad S=\pi_X(1\otimes 1).$$

In particular, we have the gauge-invariant isomorphim $C^*(A,\mathcal{V},\mathcal{H})\cong A\rtimes_X\mathbb{Z}$.

Proof. Let (π,S) be a covariant representation of $(\mathcal{V},\mathcal{H})$. One readily checks that $\pi_X(\sum_i a_i\otimes b_i):=\sum_i\pi(a_i)S\pi(b_i)$ yields an isometric linear mapping on $X_0/\|\cdot\|$ for which (2), (3) hold. Hence it extends the isometry on X such that (π,π_X) is a covariant representation of X. Conversely suppose that (π,π_X) is a covariant representation of the Hilbert bimodule X and put $S:=\pi_X(1\otimes 1)$. Then

$$S\pi(a)S^*=\pi_X((1\otimes 1)a)\pi_X(1\otimes 1)^*=\pi({}_A\langle 1\otimes a,1\otimes 1\rangle)=\pi(\mathcal{V}(a))$$

and similarly $S^*\pi(a)S=\pi_X(1\otimes 1)^*\pi_X(a(1\otimes 1))=\pi(\langle 1\otimes 1,a\otimes 1\rangle_A)=\pi(\mathcal{H}(a))$. □

By Remark 2.13 and Propositions 1.3, 2.14 we get

Proposition 2.15. *Let $\mathfrak{X}$ be the generalized C^*-correspondence constructed from $(\mathcal{V},\mathcal{H})$ in* [10, Sec. 5]. *The crossed product $C^*(A,\mathcal{V},\mathcal{H})$ of the interaction $(\mathcal{V},\mathcal{H})$ and the covariance algebra $C^*(A,\mathfrak{X})$ of $\mathfrak{X}$ are naturally isomorphic.*

2.3. **Dynamical system dual to a complete interaction.** Let $(\mathcal{V},\mathcal{H})$ be a complete interaction. Since subalgebras $\mathcal{V}(A)$ and $\mathcal{H}(A)$ are hereditary, we may, cf. [22, Prop. 4.1.9], and we will identify them with the open subsets of $\widehat{A}$: .

$$\widehat{\mathcal{V}(A)}=\{\pi\in\widehat{A}:\pi(\mathcal{V}(1))\neq 0\},\qquad \widehat{\mathcal{H}(A)}=\{\pi\in\widehat{A}:\pi(\mathcal{H}(1))\neq 0\}.$$

The isomorphisms $\mathcal{V}:\mathcal{H}(A)\to\mathcal{V}(A)$, $\mathcal{H}:\mathcal{V}(A)\to\mathcal{H}(A)$ induce (mutually inverse) homeomorphisms $\widehat{\mathcal{V}}:\widehat{\mathcal{V}(A)}\to\widehat{\mathcal{H}(A)}$ and $\widehat{\mathcal{H}}:\widehat{\mathcal{H}(A)}\to\widehat{\mathcal{V}(A)}$.

Definition 2.16. We call the partial homeomorphisms $\widehat{\mathcal{V}}:\widehat{\mathcal{V}(A)}\to\widehat{\mathcal{H}(A)}$ and $\widehat{\mathcal{H}}:\widehat{\mathcal{H}(A)}\to\widehat{\mathcal{V}(A)}$ of $\widehat{A}$ *dual to the interaction* $(\mathcal{V},\mathcal{H})$.

Remark 2.17. For an irreducible representation $\pi:A\to\mathcal{B}(H)$ with $\pi(\mathcal{H}(1))\neq 0$ representation $\widehat{\mathcal{H}}(\pi)\in\widehat{A}$ is a unique up to unitary equivalence extension of the representation

$$\widehat{\mathcal{H}}(\pi)|_{\mathcal{V}(A)}=\pi\circ\mathcal{H}:\mathcal{V}(A)\to\mathcal{B}(\pi(\mathcal{H}(1))H),$$

In the case $(\mathcal{V},\mathcal{H})$ is a C^*-dynamical system $\mathcal{H}(A)$ is an ideal and $\pi(\mathcal{H}(1))H=H$.

Proposition 2.18. *If X is the Hilbert bimodule associated to $(\mathcal{V},\mathcal{H})$ and $\widehat{h}=X$-*Ind *is the associated partial homeomorphism of $\widehat{A}$, then $\widehat{h}=\widehat{\mathcal{H}}$.*

Proof. Let $\pi:A\to\mathcal{B}(H)$ be an irreducible representation with $\pi(\mathcal{H}(1))\neq 0$. For $(a\otimes b)\otimes_\pi h\in X\otimes_\pi H$, $a,b\in A$, $h\in H$, using Proposition 2.11 iii) we have

$$\begin{aligned}\widehat{h}(\pi)(\mathcal{V}(1))(a\otimes b)\otimes_\pi h&=\big(\mathcal{V}(1)a\otimes b\big)\otimes_\pi h=\big(\mathcal{V}(1)a\mathcal{V}(1)\otimes b\big)\otimes_\pi h\\&=(1\otimes\mathcal{H}(a)b)\otimes_\pi h=(1\otimes 1)\otimes_\pi\pi(\mathcal{H}(a)b)h.\end{aligned}$$

Hence the space $H_0:=\widehat{h}(\pi)\big(\mathcal{V}(1)\big)(X\otimes_\pi H)$ is spanned by the vectors of the form $(1\otimes 1)\otimes_\pi h$, $h\in\pi(\mathcal{H}(1))H$. Moreover, since

$$\begin{aligned}\langle(1\otimes 1)\otimes_\pi h_1,(1\otimes 1)\otimes_\pi h_2\rangle&=\langle h_1,\pi(\langle 1\otimes 1,1\otimes 1\rangle_A)h_2\rangle=\langle h_1,\pi(\mathcal{H}(1))h_2\rangle\\&=\langle\pi(\mathcal{H}(1))h_1,\pi(\mathcal{H}(1))h_2\rangle\end{aligned}$$

one sees that the mapping $(1 \otimes 1) \otimes_\pi h \mapsto \pi(\mathcal{H}(1))h$ extends to a unitary operator U from H_0 onto the space $\pi(\mathcal{H}(1))H$. For $a \in \mathcal{V}(A)$ we have

$$\widehat{h}(\pi)(a)(1 \otimes 1)_\pi \otimes h = (a \otimes 1)_\pi \otimes h = (1 \otimes \mathcal{H}(a))_\pi \otimes h = (1 \otimes 1) \otimes_\pi \pi(\mathcal{H}(a))h,$$

that is $\widehat{h}(\pi)(a)U^*h = U^*\pi(\mathcal{H}(a))h$. It follows that U establishes unitary equivalence between $\widehat{h}(\pi) : \mathcal{V}(A) \to \mathcal{B}(H_0)$ and $\pi \circ \mathcal{H} : \mathcal{V}(A) \to \mathcal{B}(\pi(\mathcal{H}(1))H)$. Hence $\widehat{h} = \widehat{\mathcal{H}}$. □

Combining the above with Proposition 2.14 and Theorem 1.1 we get

Theorem 2.19. *Let $(\mathcal{V}, \mathcal{H})$ a complete interaction and $(\widehat{\mathcal{V}}, \widehat{\mathcal{H}})$ its dual.*

i) *If $\widehat{\mathcal{V}}$ is topologically free, then for every covariant representation (π, S) with π faithful the C^*-algebra generated by $\pi(A)$ and S is a copy of $C^*(A, \mathcal{V}, \mathcal{H})$*
ii) *If $\widehat{\mathcal{V}}$ is free, then $J \mapsto \widehat{J \cap A}$ is a lattice isomorphism between ideals in $C^*(A, \mathcal{V}, \mathcal{H})$ and open $\widehat{\mathcal{V}}$-invariant sets in $\widehat{A}$.*
iii) *If $\widehat{\mathcal{V}}$ is topologically free and minimal, then $C^*(A, \mathcal{V}, \mathcal{H})$ is simple.*

Actually, it follows from Propositions 2.14, 2.18, see [17, discussion before Thm. 2.5], that open $\widehat{\mathcal{V}}$-invariant sets in $\widehat{A}$ are in one-to-one correspondence with gauge invariant ideals in $C^*(A, \mathcal{V}, \mathcal{H})$. Therefore it is useful to have their description.

Lemma 2.20. *Let I be an ideal in A. The following conditions are equivalent:*

i) *The set $\widehat{I}$ is $\widehat{\mathcal{V}}$-invariant,*
ii) *$\mathcal{V}(I) \subset I$ and $\mathcal{H}(I) \subset I$,*
iii) *$\mathcal{V}(I) = \mathcal{V}(1)I\mathcal{V}(1)$.*

Proof. Let X be the Hilbert bimodule associated to $(\mathcal{V}, \mathcal{H})$ and $\widehat{h}$ its dual. It is known, see [17], that $\widehat{I}$ is $\widehat{h}$-invariant if and only $IX = XI$. If we assume $IX = XI$, then for $a \in I$, $a \otimes 1 \in XI$ and thus $\langle 1 \otimes 1, a \otimes 1\rangle_A = \mathcal{H}(a) \in I$. Hence $\mathcal{H}(I) \subset I$ and analogously one gets $\mathcal{V}(I) \subset I$. Since $\widehat{\mathcal{V}} = \widehat{h}^{-1}$ this proves the implication i)⇒ ii). Conversely, let us assume ii) and let $a \in I$, $b \in A$ and $\{\mu_\lambda\}$ be an approximate unit in I. One checks that $\lim_\lambda a\mu_\lambda \otimes b = a \otimes b$ in X. Hence using Proposition 2.11 iii) three times we get

$$a \otimes b = a\mathcal{V}(1) \otimes b = \lim_\lambda a\mathcal{V}(1)\mu_\lambda \otimes b = \lim_\lambda a\mathcal{V}(1)\mu_\lambda\mathcal{V}(1) \otimes b = \lim_\lambda a \otimes \mathcal{H}(\mu_\lambda)b \in XI.$$

It follows that $IX \subset XI$ and analogously one shows $XI \subset IX$. Thus i)⇔ ii).

ii)⇒ iii). $\mathcal{V}(I) \subset I$ implies $\mathcal{V}(I) \subset \mathcal{V}(1)I\mathcal{V}(1)$ and applying $\mathcal{V}$ to $\mathcal{H}(I) \subset I$ we get $\mathcal{V}(1)I\mathcal{V}(1) = \mathcal{V}(\mathcal{H}(I)) \subset \mathcal{V}(I)$.

iii)⇒ ii). $\mathcal{V}(I) = \mathcal{V}(1)I\mathcal{V}(1) \subset I$ and $\mathcal{H}(I) = \mathcal{H}(\mathcal{V}(1)I\mathcal{V}(1)) = \mathcal{H}(\mathcal{V}(I)) = \mathcal{H}(1)I\mathcal{H}(1) \subset I$. □

3. Graph C^*-algebras via interactions

We adhere to the notation of [20], [4]. Throughout we let $E = (E^0, E^1, r, s)$ to be a fixed finite directed graph, that is E^0 is a set of vertices, E^1 is a set of edges, $r, s : E^1 \to E^0$ are range, source maps, and we assume that both sets E_0, E^1 are finite. We write E^n, $n > 0$, for the set of paths $\mu = (\mu_1, \ldots, \mu_n)$, $r(\mu_i) = s(\mu_{i+1})$, $i = 1, ..., n-1$, of length n. The maps r, s naturally extend to E^n, so that (E^0, E^n, s, r) is the graph, and s extends to the set E^∞ of infinite paths $\mu = (\mu_1, \mu_2, \ldots)$. We also put $s(v) = r(v) = v$ for $v \in E^0$. The elements of $E^0_{sinks} := E^0 \setminus s(E^1)$ and respectively $E^0_{sources} := E^0 \setminus r(E^1)$ are called sinks and sources.

3.1. **Graph C^*-algebra $C^*(E)$ and the Bratteli diagram for its core.** In our setting a *Cuntz-Krieger E-family* compose of non-zero pair-wise orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying

$$S_e^* S_e = P_{r(e)} \quad \text{and} \quad P_v = \sum_{e \in s^{-1}(v)} S_e S_e^* \qquad \text{for all} \quad v \in s(E^1),\, e \in E^1. \tag{9}$$

One then puts $S_\mu = S_{\mu_1} S_{\mu_2} \cdots S_{\mu_n}$ for $\mu = (\mu_1, ..., \mu_n)$ ($S_\mu \neq 0 \Rightarrow \mu \in E^n$) and $S_v := P_v$ for $v \in E^0$. Relations (9) extend onto operators S_μ, see [20, Lem 1.1], as follows

$$S_\nu^* S_\mu = \begin{cases} S_{\mu'}, & \text{if } \mu = \nu\mu',\ \mu' \notin E^0, \\ S_{\nu'}^*, & \text{if } \nu = \mu\nu',\ \nu' \notin E^0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $C^*(\{P_v : v \in E^0\} \cup \{S_e : e \in E^1\}) = \overline{\operatorname{span}}\, \{S_\mu S_\nu^* : \mu \in E^n, \nu \in E^m,\, n, m \in \mathbb{N}\}$.

The *graph C^*-algebra* $C^*(E)$ of E is a universal C^*-algebra generated by a universal Cuntz-Krieger E-family $\{s_e : e \in E^1\}$, $\{p_v : v \in E^0\}$. It is equipped with the natural *circle gauge action* $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$ established by relations

$$\gamma_\lambda(p_v) = p_v, \qquad \gamma_\lambda(s_e) = \lambda s_e, \qquad v \in E^0,\, e \in E^1,\, \lambda \in \mathbb{T}. \tag{10}$$

The fixed point C^*-algebra for γ is the so-called *core* AF-algebra of the form

$$\mathcal{F}_E := \overline{\operatorname{span}}\, \{s_\mu s_\nu^* : \mu, \nu \in E^n,\ n = 0, 1, \dots\}.$$

We recall the standard Bratteli diagram for $\mathcal{F}_E$. For each vertex v and $N \in \mathbb{N}$ we set

$$\mathcal{F}_N(v) := \operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in E^N,\ r(\mu) = r(\nu) = v\},$$

which is a simple I_n factor with $n = |\{\mu \in E^N : r(\mu) = v\}|$ (if $n = 0$ we put $\mathcal{F}_N(v) := \{0\}$). The spaces

$$\mathcal{F}_N := \Big(\oplus_{v \notin E^0_{sinks}} \mathcal{F}_N(v) \Big) \oplus \Big(\oplus_{w \in E^0_{sinks}} \oplus_{i=0}^N \mathcal{F}_i(w) \Big), \qquad N \in \mathbb{N},$$

form an increasing family of finite-dimensional algebras and

$$\mathcal{F}_E = \overline{\bigcup_{N \in \mathbb{N}} \mathcal{F}_N}.$$

We denote by $\Lambda(E)$ the corresponding Bratteli diagram for $\mathcal{F}_E$. If E has no sinks we can view $\Lambda(E)$ as an infinite vertical concatenation of E where on the n-th level we have the vertices $r(E^n)$, $n = 0, 1, 2, 3, ...$, and if E has sinks one has to additionally attach to every sink an infinite tail starting from level $n = 1$, cf. [4]. We adopt the convention that if V is a subset of E^0 we treat it as a full subgraph of E and $\Lambda(V)$ stands for the corresponding Bratteli diagram for $\mathcal{F}_V$. In particular, if V is *hereditary*, i.e. $s(e) \in V \Longrightarrow r(e) \in V$ for all $e \in E^1$, and *saturated*, i.e. every vertex which feeds into V and only V is in V, then the subdiagram $\Lambda(V)$ of $\Lambda(E)$ yields an ideal in $\mathcal{F}_E$ which is naturally identified with $\mathcal{F}_V$. Roughly speaking, viewing $\Lambda(E)$ as an infinite directed graph the hereditary and saturated subgraphs (subdiagrams) of $\Lambda(E)$ correspond to ideals in $\mathcal{F}_E$, see [5, 3.3].

3.2. **$C^*(E)$ as a crossed product of a complete interaction.** For each vertex $v \in E^0$ we let $n_v := |r^{-1}(v)|$ be the number of the edges that v receives and we define an operator s in $C^*(E)$ as a sum of the partial isometries $\{s_e : e \in E^1\}$ "averaged" on the spaces corresponding to projections $\{p_v : v \in E^0 \setminus E^0_{sources}\}$:

$$s := \sum_{e \in E_1} \frac{1}{\sqrt{n_{r(e)}}}\, s_e = \sum_{v \in r(E^1)} \frac{1}{\sqrt{n_v}} \sum_{e \in r^{-1}(v)} s_e. \tag{11}$$

Then s is a partial isometry with the initial projection $s^*s = \sum_{v\in r(E^1)} p_v$ (s is an isometry iff E has no sources) and we use it to define

$$\mathcal{V}(a) := sas^*, \qquad \mathcal{H}(a) := s^*as, \qquad a \in C^*(E). \tag{12}$$

Plainly, $(\mathcal{V},\mathcal{H})$ is a complete interaction over $C^*(E)$ and $\mathcal{V}$ and $\mathcal{H}$ are unique bounded linear maps on $C^*(E)$ satisfying the following formulas

$$\mathcal{V}\Big(s_\mu s_\nu^*\Big) = \begin{cases} \frac{1}{\sqrt{n_{s(\mu)}n_{s(\nu)}}} \sum\limits_{e,f\in E^1} s_{e\mu}s_{f\nu}^*, & n_{s(\mu)}n_{s(\nu)} \neq 0,\\ 0, & n_{s(\mu)}n_{s(\nu)} = 0,\end{cases} \tag{13}$$

$$\mathcal{H}\Big(s_{e\mu}s_{f\nu}^*\Big) = \frac{1}{\sqrt{n_{s(\mu)}n_{s(\nu)}}}\, s_\mu s_\nu^*, \quad \mathcal{H}\Big(p_v\Big) = \begin{cases} \sum\limits_{e\in s^{-1}(v)} \frac{p_{r(e)}}{n_{r(e)}}, & v \notin E^0_{sinks},\\ 0, & v\in E^0_{sinks},\end{cases} \tag{14}$$

where $\mu\in E^n$, $\nu\in E^m$, $n,m\in\mathbb{N}$, $e,f\in E^1$, $v\in E^0$. We note that, even though $\mathcal{H}$ always does, $\mathcal{V}$ hardly ever preserves the canonical MASA $\mathcal{D}_E := \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu\in E^n, n\in\mathbb{N}\}$ in $\mathcal{F}_E$. Fortunately, both $\mathcal{V}$ and $\mathcal{H}$ invariate $\mathcal{F}_E$ (which follows immediately from (13), (14)).

Definition 3.1. We say that the pair $(\mathcal{V},\mathcal{H})$ of continuous maps on $\mathcal{F}_E$ satisfying (13), (14) is the (complete) *interaction on $\mathcal{F}_E$ associated to the graph E.*

We may use Proposition 2.8 to determine when the interaction $(\mathcal{V},\mathcal{H})$ is a C^*-dynamical system. In particular, this is always the case when E has no sources.

Proposition 3.2. *The interaction $(\mathcal{V},\mathcal{H})$ associated to E is a C^*-dynamical system if and only if every two paths with the same length and ending either both starts in sources or not in sources.*

Proof. We recall that $\mathcal{H}(1) = s^*s = \sum_{v\in r(E^1)} p_v$. In view of the equivalence i) $\Leftrightarrow$ iii) in Proposition 2.8 the equivalence in the present assertion follows from the relations

$$\mathcal{H}(1)s_\mu s_\nu^* = \begin{cases} 0, & \text{if } s(\mu)\notin r(E^1)\\ s_\mu s_\nu^*, & \text{otherwise}\end{cases}, \quad s_\mu s_\nu^*\mathcal{H}(1) = \begin{cases} 0, & \text{if } s(\nu)\notin r(E^1)\\ s_\mu s_\nu^*, & \text{otherwise}\end{cases}.$$

□

A natural question to ask is when $\mathcal{H}$ is multiplicative, but it is hardly the case.

Proposition 3.3. *The pair $(\mathcal{H},\mathcal{V})$ where $(\mathcal{V},\mathcal{H})$ is the interaction associated to E is a C^*-dynamical system if and only if the mapping $r : E^1 \to E^0$ is injective.*

Proof. By Proposition 2.8 multiplicativity of $\mathcal{H}$ is equivalent to $\mathcal{V}(1)$ being a central element in $\mathcal{F}_E$. If $r : E^1\to E^0$ is injective, then $\mathcal{F}_E = \mathcal{D}_E$ is commutative and $(\mathcal{H},\mathcal{V})$ is a C^*-dynamical system because $\mathcal{V}(1)\in\mathcal{F}_E$. Conversely, if we assume that the element $\mathcal{V}(1) = ss^* = \sum\limits_{v\in r(E^1)} \frac{1}{n_v} \sum\limits_{e,f\in r^{-1}(v)} s_e s_f^*$ is central, then for all $g,h\in E^1$ such that $r(g) = r(h) = v$

$$\frac{1}{n_v}\sum_{e\in r^{-1}(v)} s_e s_h^* = \mathcal{V}(1)s_g s_h^* = s_g s_h^*\mathcal{V}(1) = \frac{1}{n_v}\sum_{f\in r^{-1}(v)} s_g s_f^*.$$

This implies that $r^{-1}(v) = \{g\} = \{h\}$ and hence $r : E^1\to E^0$ is injective. □

The main goal of the present subsection is

Theorem 3.4. *We have a one-to-one correspondence between Cuntz-Krieger E-families $\{P_v : v\in E^0\}$, $\{S_e : e\in E^1\}$ for E and faithful covariant representations (π, S) of the interaction $(\mathcal{V},\mathcal{H})$ associated to E. It is given by the relations*

$$S = \sum_{e\in E_1} \frac{1}{\sqrt{n_{r(e)}}} S_e, \qquad P_v = \pi(p_v), \qquad S_e = \sqrt{n_{r(e)}}\,\pi(s_e s_e^*) S.$$

In particular, we have a natural isomorphism $C^*(E) \cong C^*(\mathcal{F}_E, \mathcal{V}, \mathcal{H})$.

Proof. A Cuntz-Krieger E-family $\{P_v : v \in E^0\}$, $\{S_e : e \in E^1\}$ yields a representation π of $C^*(E)$ which is well known to be faithful on $\mathcal{F}_E$. By the definition of $(\mathcal{V}, \mathcal{H})$ the pair $(\pi|_{\mathcal{F}_E}, S)$ where $S := \pi(s) = \sum_{e \in E_1} \frac{1}{\sqrt{n_{r(e)}}} S_e$ is a covariant representation of $(\mathcal{V}, \mathcal{H})$. Conversely, let (π, S) be a faithful representation of $(\mathcal{V}, \mathcal{H})$ and put $P_v := \pi(p_v)$ and $S_e := \sqrt{n_{r(e)}}\pi(s_e s_e^*)S$. We claim that $\{P_v : v \in E^0\}$, $\{S_e : e \in E^1\}$ is a Cuntz-Krieger E-family such that $S = \sum_{e \in E_1} \frac{S_e}{\sqrt{n_{r(e)}}}$. Indeed, for $e \in E^1$ we have

$$S_e^* S_e = n_{r(e)} \pi(p_{r(e)}) \pi(\mathcal{H}(s_e s_e^*)) \pi(p_{r(e)}) = \pi(p_{r(e)}) = P_{r(v)},$$

and for $v \in s(E^1)$

$$\begin{aligned}\sum_{e \in s^{-1}(v)} S_e S_e^* &= \sum_{e \in s^{-1}(v)} n_{r(e)} \pi(s_e s_e^*) \pi(\mathcal{V}(1)) \pi(s_e s_e^*) \\ &= \sum_{e \in s^{-1}(v), e_1, e_2 \in E^1} \frac{n_{r(e)}}{\sqrt{n_{r(e_1)} n_{r(e_2)}}} \pi(s_e s_e^* (s_{e_1} s_{e_2}^*) s_e s_e^*) \\ &= \sum_{e \in s^{-1}(v)} \pi(s_e s_e^*) = \pi(p_v) = P_v.\end{aligned}$$

Now note that $S^*S = \pi(\mathcal{H}(1)) = \sum_{v \in r(E^1)} \pi(p_v)$ and therefore $S = \sum_{e \in E^1} S\pi(p_v)$. Moreover, for each $v \in r(E^1)$ we have

$$\begin{aligned}\left(\sum_{e \in r^{-1}(v)} \pi(s_e s_e^*)\right) S\pi(p_v) S^* &= \sum_{e \in r^{-1}(v)} \pi(s_e s_e^* \mathcal{V}(p_v)) = \sum_{e, e_1, e_2 \in r^{-1}(v)} \frac{\pi(s_e s_e^* s_{e_1} s_{e_2}^*)}{n_v} \\ &= \sum_{e_1, e_2 \in r^{-1}(v)} \frac{\pi(s_{e_1} s_{e_2}^*)}{n_v} = \pi(\mathcal{V}(p_v)) = S\pi(p_v) S^*.\end{aligned}$$

Hence the final space of the partial isometry $S\pi(p_v)$ decomposes into the orthogonal sum of ranges of the projections $\pi(S_e S_e^*)$, $e \in r^{-1}(v)$, and consequently

$$\sum_{e \in E^1} \frac{S_e}{\sqrt{n_{r(e)}}} = \sum_{e \in E^1} \pi(s_e s_e^*) S\pi(p_{r(e)}) = \sum_{v \in E^0} \sum_{e \in r^{-1}(v)} \pi(s_e s_e^*) S\pi(p_v) = S.$$

□

Remark 3.5. If E has no sources, then s is an isometry and $\mathcal{V}$ is a monomorphism (with hereditary range). In this case $C^*(E)$ coincides with various crossed products by endomorphisms that involve isometries, cf. [2], and in particular, one reconciles an intersection of Theorem 3.4 with [15, Thm. 5.2] (proved for locally finite graphs without sources).

The canonical completely positive mapping $\phi_E : C^*(E) \to C^*(E)$ is given by the formula

$$\phi_E(x) = \sum_{e \in E^1} s_e x s_e^*,$$

and this map (unlike $\mathcal{V}$ but like $\mathcal{H}$) preserves both $\mathcal{F}_E$ and $\mathcal{D}_E$. Moreover, $(\phi_E, \mathcal{H})$ forms a C^*-dynamical on $\mathcal{D}_E$ and if E has no sinks the same relations as in Theorem 3.4 yield an isomorphism between $C^*(E)$ and the Exel's crossed product $\mathcal{D}_E \rtimes_{(\phi_E, \mathcal{H})} \mathbb{N}$, see [7, Thm. 5.1]. The advantage of $\mathcal{D}_E \rtimes_{(\phi_E, \mathcal{H})} \mathbb{N}$ over $C^*(\mathcal{F}_E, \mathcal{V}, \mathcal{H})$ is that it starts from a commutative C^*-algebra $\mathcal{D}_E$. The disadvantages are that the dynamics in $(\phi_E, \mathcal{H})$ is irreversible and involves two mappings, while in essence $(\mathcal{V}, \mathcal{H})$ is a single map, cf. Proposition 2.9.

3.3. **Powers of the partial isometry s and iterates of the interaction associated to E.** The following discussion serves as an interesting example illustrating the general (mis)behavior of the iterates of interactions and C^*-dynamical systems.

In [14] Halmos and Wallen presented a method of constructing an operator S such that the distribution of values of n for which S^n is or is not a partial isometry is arbitrary. En passant we discover a similar construction based on graphs. To this end we use a partially-stochastic matrix $P = [p_{v,w}]$ arising from the adjacency matrix $A = [A(v,w)]_{v,w\in E^0}$ of the graph E. Namely, we let

$$p_{v,w} := \begin{cases} \frac{A(v,w)}{n_w}, & A(v,w) \neq 0, \\ 0, & A(v,w) = 0, \end{cases} \tag{15}$$

where $A(v,w) = |\{e \in E^1 : s(e) = v, r(e) = w\}|$ and by a *partially-stochastic matrix* we mean a non-negative matrix in which each non-zero column sums up to one.

Proposition 3.6. *Let s be the operator given by* (11) *and let $n \geq 1$. The following conditions are equivalent:*

i) *operator s^n is a partial isometry,*
ii) *n-th power of the matrix $P = \{p_{v,w}\}_{v,w\in E^0}$ is partially-stochastic,*
iii) *for any $\mu \in E^n$ there is no $\nu \in E^k$ such that $r(\mu) = r(\nu)$, $k < n$ and $s(\nu) \in E^0_{sources}$,*
iv) *$(\mathcal{V}^n, \mathcal{H}^n)$ is an interaction over $\mathcal{F}_E$.*
v) *$(\phi^n_E, \mathcal{H}^n)$ is a C^*-dynamical system on $\mathcal{D}_E$.*

Proof. To see the equivalence i) $\Leftrightarrow$ ii) note that s^n is a partial isometry if and only if $s^{*n}s^n = \mathcal{H}^n(1)$ is an orthogonal projection. Moreover, since $\mathcal{H}(p_v) = \sum_{w\in E^0} p_{v,w}p_w$, cf. (14), we get

$$\mathcal{H}^n(1) = \sum_{v_0,\dots,v_n\in E^0} p_{v_0,v_1}\cdot p_{v_1,v_2}\cdot \ldots \cdot p_{v_{n-1},v_n} p_{v_n} = \sum_{v,w\in E^0} p^{(n)}_{v,w} p_w$$

where $P^n = \{p^{(n)}_{v,w}\}_{v,w\in E^0}$ stands for the n-th power of P. By the orthogonality of projections p_w, it follows that $\mathcal{H}^n(1)$ is a projection iff $\sum_{v\in E^0} p^{(n)}_{v,w} \in \{0,1\}$ for all $w \in E^0$, that is iff P^n is partially-stochastic. This proves i) $\Leftrightarrow$ ii). To show ii) $\Leftrightarrow$ iii) note that the condition $\sum_{v\in E^0} p^{(n)}_{v,w} > 0$ is equivalent to the existence of $\mu \in E^n$ such that $w = r(\mu)$. Thus if iii) is not true there is $w \in E^0$ and $v_0 \in E^0_{sources}$ such that $\sum_{v\in E^0} p^{(n)}_{v,w} > 0$ and $p^{(k)}_{v_0,w} > 0$ for $k < n$. Then $\sum_{v\in E^0} p^{(n-k)}_{v,v_0} = 0$ (because $v_0 \in E^0_{sources}$) and therefore

$$0 < \sum_{v\in E^0} p^{(n)}_{v,w} = \sum_{v,v_{n-k}\in E^0} p^{(n-k)}_{v,v_{n-k}} p^{(k)}_{v_{n-k},w} \leq \sum_{v_{n-k}\in E^0\setminus\{v_0\}} p^{(k)}_{v_{n-k},w} < 1.$$

Hence P^n is not partially-stochastic. Conversely, we assume iii) and $\sum_{v\in E^0} p^{(n)}_{v,w} > 0$. Then $p^{(n-k)}_{v_k,w} \neq 0$ for $0 < k < n$ implies that $v \notin E^0_{sources}$ and consequently $\sum_{v_{k-1}\in E^0} p^{(1)}_{v_{k-1},v_k} = 1$. Therefore (by induction on k)

$$\sum_{v\in E^0} p^{(n)}_{v,w} = \sum_{v_0,v_1\in E^0} p^{(1)}_{v_0,v_1} p^{(n-1)}_{v_1,w} = \sum_{v_1\in E^0} p^{(n-1)}_{v_1,w} = \ldots = \sum_{v_{n-1}\in E^0} p^{(1)}_{v_{n-1},w} = 1.$$

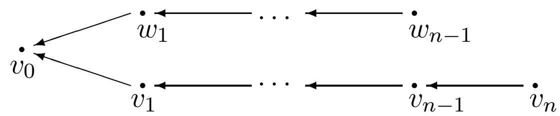
This proves ii). Now as $\mathcal{V}^n(\cdot) = s^n(\cdot)s^{*n}$ and $\mathcal{H}^n(\cdot) = s^{*n}(\cdot)s^n$ one readily sees that i) implies iv) and if we assume iv) then s^n is a partial isometry because $\mathcal{H}^n(1)$ is a projection by Lemma 2.5. Thus i) $\Leftrightarrow$ iv).
To see ii) $\Leftrightarrow$ v) note that the iteration of an endomorphism and its transfer operator gives again an endomorphism and its transfer operator. Thus $(\phi^n_E, \mathcal{H}^n)$ is a C^*-dynamical system

iff the transfer operator $\mathcal{H}^n$ is non-degenerate, that is iff $\phi_E^n(\mathcal{H}^n(1)) = \phi_E^n(1)$. Since

$$\phi_E^n(\mathcal{H}^n(1)) = \sum_{v,w\in E^0} p_{v,w}^{(n)}\phi_E^n(p_w) = \sum_{v\in E^0,\mu\in E^n} p_{v,r(\mu)}^{(n)} s_\mu s_\mu^* \quad \text{and} \quad \phi_E^n(1) = \sum_{\mu\in E^n} s_\mu s_\mu^*$$

we get that $\phi_E^n(\mathcal{H}^n(1)) = \phi_E^n(1)$ iff $P^n = \{p_{v,w}^{(n)}\}_{v,w\in E^0}$ is partially-stochastic. □

Example 3.7. The partial isometry s arising from the following graph



has the property that the only power of s which is not a partial isometry is the n-th one. Hence by considering a disjoint sum of the above graphs for a chosen sequence of natural numbers $1 < n_1 < n_2 < ... < n_m$ one obtains a partial isometry whose k-th power is a partial isometry iff $k \neq n_i$, $i = 1, ..., m$.

We may use Proposition 3.6 to prolong the list of equivalents in Proposition 3.2, see also Proposition 2.8.

Corollary 3.8. *Let $(\mathcal{V}, \mathcal{H})$ be the interaction associated to E. The following conditions are equivalent:*

i) $(\mathcal{V}, \mathcal{H})$ *is a C^*-dynamical system,*
ii) *every power of the matrix $P = \{p_{v,w}\}_{v,w\in E^0}$ is partially-stochastic,*
iii) *operator s given by* (11) *is a power partial isometry,*
iv) $(\mathcal{V}^n, \mathcal{H}^n)$ *is an interaction for all $n \in \mathbb{N}$.*
v) $(\phi_E^n, \mathcal{H}^n)$ *is a C^*-dynamical system for all $n \in \mathbb{N}$.*

Proof. It suffices to note that the condition described in the assertion of Proposition 3.2 holds if and only if item iii) in Proposition 3.6 holds for every n. □

3.4. **Description of the dynamical system dual to $(\mathcal{V}, \mathcal{H})$.** We obtain a quite satisfactory picture of the system $(\widehat{\mathcal{V}}, \widehat{\mathcal{H}})$ dual to the interaction $(\mathcal{V}, \mathcal{H})$ associated to E using the one-sided Markov shift (Ω_E, σ_E) where $\Omega_E = \bigcup_{N=0}^\infty E_{sinks}^N \cup E^\infty$, $E_{sinks}^N = \{\mu \in E^N : r(\mu) \in E_{sinks}^0\}$ and σ_E is defined on $\Omega_E \setminus E_{sinks}^0$ via the formula

$$\sigma_E(\mu_1, \mu_2, \mu_3...) = (\mu_2, \mu_3...) \qquad \text{for } (\mu_1, \mu_2...) \in \bigcup_{N=2}^{\infty} E_{sinks}^N \cup E^\infty,$$

and $\sigma_E(\mu) = r(\mu)$ for $\mu \in E_{sinks}^1$. Equipped with a natural topology, cf. e.g. [16, Lem. 3.2], Ω_E is a compact space and there is an isomorphism $\mathcal{D}_E \cong C(\Omega)$ which intertwines $\phi_E : \mathcal{D}_E \to \mathcal{D}_E$ with the transpose to σ_E.

To start with we note that the infinite direct sum $\oplus_{N=0}^\infty \oplus_{w\in E_{sinks}^0} \mathcal{F}_N(w)$ yields an ideal I_{sinks} in $\mathcal{F}_E$ generated by the projections p_w, $w \in E_{sinks}^0$. We rewrite it in the form

$$I_{sinks} = \bigoplus_{N\in\mathbb{N}} G_N, \qquad \text{where} \quad G_N := \Big(\oplus_{w\in E_{sinks}^0} \mathcal{F}_N(w)\Big).$$

If $\mathcal{F}_N(w) \neq \{0\}$ (since it is a finite factor) there is up to equivalence a unique irreducible representation $\pi_{w,N}$ of $\mathcal{F}_E$ such that $\ker \pi_{w,N} \cap \mathcal{F}_N(w) = \{0\}$. Consequently,

$$\widehat{G}_N = \{\pi_{w,N} : \text{ there is } \mu \in E_{sinks}^N \text{ such that } r(\mu) = w\}.$$

The complement of $\widehat{I}_{sinks} = \bigcup_{N=0}^\infty \widehat{G}_N$ in $\widehat{\mathcal{F}_E}$ is a closed set which one usually identifies with the spectrum of the quotient algebra

$$G_\infty := \mathcal{F}_E / I_{sinks}.$$

We will describe a dense subset of $\widehat{G}_\infty$ that arises from states generalizing Glimm's product states for UHF-algebras, cf. e.g. [22, 6.5]. We use the following equivalence relation on E^∞:

$$\mu \sim \nu \iff \text{there exists } N \text{ such that } (\mu_N, \mu_{N+1}, ...) = (\nu_N, \nu_{N+1}, ...).$$

In [8] the equivalence class for μ is denoted by $W(\mu)$ and is referred to as an *unstable manifold* of μ. We will slightly change the meaning of this notation. Namely, we treat μ as a subdiagram of $\Lambda(E)$ (where the only vertex on the n-th level is $s(\mu_n)$) and denote by $W(\mu)$ the full subdiagram of $\Lambda(E)$ consisting of all ancestors of the vertices that form $\mu \subset \Lambda(E)$.

Proposition 3.9. *For any infinite path $\mu \in E^\infty$ the formula*

$$\omega_\mu(s_\nu s_\eta^*) = \begin{cases} 1 & \nu = \eta = (\mu_1, ..., \mu_n) \\ 0 & \textit{otherwise} \end{cases}, \qquad \textit{for} \quad \nu, \eta \in E^n, \tag{16}$$

determines a pure state $\omega_\mu : \mathcal{F}_E \to \mathbb{C}$ (a pure extension of the point evaluation δ_μ acting on the masa $\mathcal{D}_E = C(\Omega_E)$). Moreover, denoting by π_μ the GNS-representation associated to ω_μ we have $\pi_\mu \in \widehat{G}_\infty$ and

i) *the complement of the subdiagram of the Bratteli diagram $\Lambda(E)$ corresponding to $\ker \pi_\mu$ is $W(\mu)$,*

ii) *representations π_μ and π_ν are unitarily equivalent if and only if $\mu \sim \nu$.*

Proof. The functional ω_μ is a pure state on each $\mathcal{F}_k$, $k \in \mathbb{N}$, and thus it is also a pure state on $\mathcal{F}_E = \overline{\bigcup_{k\in\mathbb{N}} \mathcal{F}_k}$, cf. [5, 4.16]. Item i) follows from the form of primitive ideal subdiagrams, see [5, 3.8], and the fact that $\ker \pi_\mu$ is the largest ideal contained in $\ker \omega_\mu$. To show item ii) note that if $(\mu_{N+1}, \mu_{N+2}, ...) = (\nu_{N+1}, \nu_{N+2}, ...)$, then both $s_{\mu_1...\mu_N} s_{\mu_1...\mu_N}^*$ and $s_{\nu_1,...,\nu_N} s_{\nu_1,...,\nu_N}^*$ are in $F_N(v)$ where $v = r(\mu_N)$ and since $F_N(v) \cong M_n(\mathbb{C})$ there is a unitary $u \in F_N(v)$ such that $\omega_\mu(a) = \omega_\nu(u^*au)$ for $a \in F_N(v)$. Then automatically $\omega_\mu(a) = \omega_\nu(u^*au)$ for all $a \in \mathcal{F}_E$ and hence $\pi_\mu \cong \pi_\nu$. Conversely, suppose that $\pi_\mu \cong \pi_\nu$, then, cf. [22, 3.13.4], there is a unitary $u \in \mathcal{F}_E$ such that $\omega_\mu(a) = \omega_\nu(u^*au)$ for all $a \in \mathcal{F}_E$. For sufficiently large n there is $x \in \mathcal{F}_n$ with $\|u - x\| < \frac{1}{2}$ and $\|x\| \leq 1$. To get the contradiction we assume that $\mu_k \neq \nu_k$ for some $k > n$. The element $a := s_{\mu_1...\mu_k} s_{\mu_1...\mu_k}^* \in \mathcal{F}_k$ commutes with all the elements from $\mathcal{F}_n$. Indeed, if $b = s_\alpha s_\beta^* \in \mathcal{F}_n$, $\alpha, \beta \in E^n$, then either $s_{\alpha f} s_{\beta f}^* = 0$ for all $f \in E^{k-n}$ and then $ab = ba = 0$ or $b = \sum_{f\in E^{k-n}} s_{\alpha f} s_{\beta f}^*$ and then $ab = s_{\alpha\mu_{n+1}...\mu_k} s_{\beta\mu_{n+1}...\mu_k}^* = ba$. From this it also follows that $\omega_\nu(ab) = 0$ for all $b \in \mathcal{F}_n$. Accordingly, $\omega_\nu(x^*ax) = \omega_\nu(ax^*x) = 0$ and since $\|(u^* - x^*)au\| = \|x^*a(u-x)\| < 1/2$ we get $1 > \omega_\nu((u^* - x^*)au) + \omega_\nu(x^*a(u-x)) = \omega_\nu(u^*au) - \omega_\nu(x^*ax) = \omega_\mu(a) = 1$, an absurd. □

Theorem 3.10. *Under the above notation the space $\widehat{\mathcal{F}_E}$ admits the following decomposition into disjoint sets*

$$\widehat{\mathcal{F}_E} = \bigcup_{N=0}^{\infty} \widehat{G}_N \cup \widehat{G}_\infty$$

where $\widehat{G}_N$ are open discrete and $\widehat{G}_\infty$ is a closed subset of $\widehat{\mathcal{F}_E}$. The set

$$\Delta = \widehat{\mathcal{F}_E} \setminus \widehat{G}_0$$

is the domain of $\widehat{\mathcal{V}}$ which acts on the corresponding representations as follows:

$$\widehat{\mathcal{V}}(\pi_{(\mu_1,\mu_2,\mu_3,...)}) = \pi_{(\mu_2,\mu_3,...)}, \qquad \textit{for } (\mu_1, \mu_2, \mu_3, ...) \in E^\infty,$$

$$\widehat{\mathcal{V}}(\pi_{w,N}) = \pi_{w,N-1}, \qquad \textit{for } w = r(\mu) \textit{ where } \mu \in E^N_{sinks},\ N \geq 1.$$

In particular, $\pi_{w,N} \in \widehat{\mathcal{V}}(\Delta)$ iff there is $\mu \in E^{N+1}_{sinks}$ such that $r(\mu) = w$, and then $\widehat{\mathcal{H}}(\pi_{w,N}) = \pi_{w,N+1}$. Similarly, $\pi_\mu \in \widehat{\mathcal{V}}(\Delta)$ iff there is $\nu \sim \mu$ such that $s(\nu)$ is not a source, and then for $\nu_0 \in E^1$ such that $(\nu_0, \nu_1, \nu_2, ...) \in E^\infty$ we have $\widehat{\mathcal{H}}(\pi_\mu) = \pi_{(\nu_0,\nu_1,\nu_2,...)}$.

Proof. The first part of the assertion follows from the construction of the sets $\widehat{G}_N$, $\widehat{G}_\infty$. To see that $\widehat{\mathcal{V}(\mathcal{F}_E)} = \{\pi \in \widehat{\mathcal{F}_E} : \pi(\mathcal{V}(1)) \neq 0\}$ coincides with $\Delta = \widehat{\mathcal{F}_E} \setminus \widehat{G}_0$ let $\pi \in \widehat{\mathcal{F}_E}$ and note that

$$\pi(\mathcal{V}(1)) = 0 \Longleftrightarrow \forall_{v\in s(E^1)}\, \pi(p_v) = 0 \Longleftrightarrow \exists_{w\in E^0_{sinks}} \pi \cong \pi_{w,0}.$$

Furthermore, by (13) and (14), for $N \in \mathbb{N}$ we have

$$\mathcal{V}(\mathcal{F}_N(v)) = \mathcal{V}(1)\mathcal{F}_{N+1}(v)\mathcal{V}(1), \quad \mathcal{H}(\mathcal{F}_N(v)) = \begin{cases} \mathcal{F}_{N-1}(v), & N > 0, \\ \sum\limits_{w\in r(s^{-1}(v))} \mathcal{F}_0(w) & N = 0. \end{cases} \tag{17}$$

In particular, for $N > 0$, $\pi_{w,N} \in \Delta$ and

$$(\pi_{w,N} \circ \mathcal{V})(\mathcal{F}_{N-1}(w)) = \pi_{w,N}(\mathcal{V}(1)\mathcal{F}_N(w)\mathcal{V}(1)) \neq 0.$$

Hence $\widehat{\mathcal{V}}(\pi_{w,N}) \cong \pi_{w,N-1}$. Let us now fix $\mu = (\mu_1, \mu_2, \mu_3, ...) \in E^\infty$. Let $\pi_\mu : \mathcal{F}_E \to H_\mu$ be the representation and $\xi_\mu \in H_\mu$ the cyclic vector associated to the pure state ω_μ given by (16). For $\nu, \eta \in E^n$, using (13) and (16), we get

$$\omega_\mu(\mathcal{V}(s_\nu s_\eta^*)) = \begin{cases} \frac{1}{\sqrt{n_{s(\nu)}n_{s(\eta)}}} \sum\limits_{e,f\in E^1} \omega_\mu(s_{e\nu}s_{f\eta}^*), & n_{s(\nu)}n_{s(\eta)} \neq 0, \\ 0, & n_{s(\nu)}n_{s(\eta)} = 0, \end{cases}$$

$$= \begin{cases} \frac{1}{n_{r(\mu_1)}}, & \nu = \eta = (\mu_2, ..., \mu_{n+1}) \\ 0 & \text{otherwise} \end{cases} = \frac{1}{n_{r(\mu_1)}}\, \omega_{\sigma_E(\mu)}(s_\nu s_\eta^*).$$

Hence $\omega_\mu \circ \mathcal{V} = \frac{1}{n_{r(\mu_1)}}\, \omega_{\sigma_E(\mu)}$ and therefore $\widehat{\mathcal{V}}(\pi_\mu) \cong \pi_{\sigma_E(\mu)}$, cf. [22, Cor. 3.3.8]. □

Remark 3.11. If we extend the equivalence relation $\sim$ from E^∞ onto the whole space Ω_E, defining it for $\mu \in E^N_{sinks}$ as follows

$$\mu \sim \nu \quad \Longleftrightarrow \quad \nu \in E^N_{sinks} \quad \text{and} \quad r(\mu_N) = r(\nu_N),$$

then Theorem 3.10 states that the quotient system $(\Omega_E/\sim, \sigma_E/\sim)$ is a subsystem of $(\widehat{\mathcal{F}_E}, \widehat{\mathcal{V}})$ and the relation $\sim$ coincides with the unitary equivalence of GNS-representations associated to pure extensions of the pure states of $\mathcal{D}_E = C(\Omega_E)$.

Remark 3.12. The nontrivial dynamics of the system $(\widehat{\mathcal{F}_E}, \widehat{\mathcal{V}})$ takes place in the subsystem $(\widehat{G}_\infty, \widehat{\mathcal{V}})$ where G_∞ is a C^*-algebra arising from a graph which has no sinks. Indeed, the saturation $\overline{E^0}_{sinks}$ of E^0_{sinks} (the minimal saturated set containing E^0_{sinks}) is the hereditary and saturated set corresponding to the ideal I_{sinks} in $\mathcal{F}_E$. Hence $I_{sinks} = \mathcal{F}_{\overline{E^0}_{sinks}}$ and

$$G_\infty \cong \mathcal{F}_{E^0_{sinkless}} \quad \text{where} \quad E^0_{sinkless} := E^0 \setminus \overline{E^0}_{sinks}.$$

3.5. **Identification of the properties of the dual map $\widehat{\mathcal{V}}$.** The condition (L) introduced in [20] requires that every loop in E has an exit. For convenience, by loops we will mean simple loops, that is paths $\mu = (\mu_1, ..., \mu_n)$ such that $s(\mu_1) = r(\mu_n)$ and $s(\mu_1) \neq r(\mu_k)$, for $k = 1, ..., n-1$. A loop μ is said to have an exit if it is connected to a vertex not lying on μ.

Proposition 3.13. *If every loop in E has an exit, then every nonempty open set in $\widehat{G}_\infty$ contains uncountably many non-periodic points for $\widehat{\mathcal{V}}$ (in particular, $\widehat{\mathcal{V}}$ is topologically free).*

Proof. By Remark 3.12 we may assume $G_\infty = \mathcal{F}_E$, i.e. E has no sinks. Any nonempty open set in $\widehat{\mathcal{F}_E}$ is of the form $\widehat{J} = \{\pi \in \widehat{\mathcal{F}_E} : \ker\pi \not\supseteq J\}$ where J is a non-zero ideal in $\mathcal{F}_E$. Equivalently, in terms of Bratteli diagrams

$$\widehat{J} = \{\pi \in \widehat{\mathcal{F}_E} : \Lambda(J) \setminus \Lambda(\ker\pi) \neq \emptyset\}$$

where $\Lambda(K)$ stands for the Bratteli diagram of an ideal K in $\mathcal{F}_E$. Since E is finite, without sinks, every loop in E has an exit and $\Lambda(J)$ contains all its descendants, there must be

a vertex v which appears in $\Lambda(J)$ infinitely many times and which is a base point of two different loops say μ^0 and μ^1. Suppose that v appears on the 0-th level of $\Lambda(J)$. Writing $\mu^\epsilon = \mu^{\epsilon_1}\mu^{\epsilon_2}\mu^{\epsilon_3}... \in E^\infty$ for $\epsilon = \{\epsilon_i\}_{i=1}^\infty \in \{0,1\}^{\mathbb{N}\setminus\{0\}}$ one has $\Lambda(J)\setminus\Lambda(\ker\pi_{\mu^\epsilon}) = \Lambda(J)\cap W(\mu^\epsilon) \neq \emptyset$. Moreover, by Proposition 3.9 ii), $\pi_{\mu^\epsilon} \cong \pi_{\mu^{\epsilon'}}$ if and only if ϵ and ϵ' eventually coincide. Since there is an uncountable number of non-periodic sequences in $\{0,1\}^{\mathbb{N}\setminus\{0\}}$ which pair-wisely do not eventually coincide, by Theorem 3.10, the paths corresponding to these sequences give rise to the uncountable family of non-equivalent non-periodic representations π_μ in $\widehat{J}$. □

Example 3.14. If E is the graph with a single vertex and n edges, then $C^*(E) = \mathcal{O}_n$ is the Cuntz algebra and $\mathcal{F}_E$ is an UHF-algebra. Thus $\mathrm{Prim}\,(\mathcal{F}_E) = \{0\}$ and the Rieffel homeomorphism given by $X = \mathcal{F}_E s \mathcal{F}_E$, cf. Propositions 2.14, 2.18, is not topologically free on $\mathrm{Prim}\,(\mathcal{F}_E)$. However, $\widehat{\mathcal{F}_E}$ is uncountable and X-Ind $= \widehat{\mathcal{V}}^{-1}$ is topologically free on $\widehat{\mathcal{F}_E}$.

Suppose now that μ is a loop in E. Let $\mu^\infty \in E^\infty$ be the path obtained by the infinite concatenation of μ. Then $\Lambda(E) \setminus W(\mu^\infty)$ is a Bratteli diagram for a primitive ideal in $\mathcal{F}_E$, which we denote by I_μ. Actually, by Proposition 3.9 i) we have

$$I_\mu = \ker \pi_{\mu^\infty}$$

where π_{μ^∞} is the irreducible representation associated to μ^∞.

Proposition 3.15. *If the loop μ has no exits, then up to unitary equivalence π_{μ^∞} is the only representation of $\mathcal{F}_E$ whose kernel is I_μ and the singleton $\{\pi_{\mu^\infty}\}$ is an open set in $\widehat{\mathcal{F}}_E$.*

Proof. The quotient $\mathcal{F}_E/I_\mu$ is an AF-algebra with the diagram $W(\mu^\infty)$. The path μ^∞ treated as a subdiagram of $W(\mu^\infty)$ is hereditary and its saturation $\overline{\mu^\infty}$ yields an ideal $\mathcal{K}$ in $\mathcal{F}_E/I_\mu$. Since μ^∞ has no exits, $\mathcal{K}$ is isomorphic to the ideal of compact operators $\mathcal{K}(H)$ on a separable Hilbert space H (finite or infinite dimensional). Therefore every faithful irreducible representation of $\mathcal{F}_E/I_\mu$ is unitarily equivalent to the unique extension of the isomorphism $\mathcal{K} \cong \mathcal{K}(H)$. This shows that π_{μ^∞} is determined by its kernel. Moreover, the subdiagram $\overline{\mu^\infty}$ is hereditary and saturated not only in $W(\mu^\infty)$ but also in $\Lambda(E)$. Thus we let now $\mathcal{K}$ stand for the ideal corresponding to $\overline{\mu^\infty}$ in $\mathcal{F}_E$. Let P be a primitive ideal in $\mathcal{F}_E$. As $\mathcal{K}$ is simple $P \not\supseteq \mathcal{K}$ implies $\mathcal{K} \cap P = \{0\}$. By the form of $W(\mu^\infty)$ and hereditariness of $\Lambda(P)$, $\mathcal{K} \cap P = \{0\}$ implies $\Lambda(P) \subset \Lambda(\mathcal{F}_E) \setminus W(\mu^\infty) = \Lambda(I_\mu)$. However, if $P \subset I_\mu$, we must have $P = I_\mu$ because no part of $\Lambda(I_\mu)$ is not connected to $W(\mu^\infty)$ (consult the form of diagrams of primitive ideals [5, 3.8]). Concluding, we get

$$\{P \in \mathrm{Prim}\,(\mathcal{F}_E) : P \not\supseteq \mathcal{K}\} = \{P \in \mathrm{Prim}\,(\mathcal{F}_E) : \mathcal{K} \cap P = \{0\}\} = \{I_\mu\},$$

that is $\{I_\mu\}$ is open in $\mathrm{Prim}\,(\mathcal{F}_E)$ and $\widehat{\mathcal{K}} = \{\pi_{\mu^\infty}\}$ is open in $\widehat{\mathcal{F}}_E$. □

We have the following characterizations of minimality of $\widehat{\mathcal{V}}$.

Proposition 3.16. *The map $V \mapsto \widehat{\mathcal{F}_{\Lambda(V)}}$ is a one-to-one correspondence between the hereditary saturated subsets of E^0 and open invariant sets for $\widehat{\mathcal{V}}$. In particular, $\widehat{\mathcal{V}}$ is minimal if and only if there are no non-trivial hereditary saturated subsets of E^0.*

Proof. It suffices to show that the map $V \mapsto \Lambda(V)$ is a one-to-one correspondence between the hereditary saturated subset of E^0 and Bratteli diagrams for ideals in $\mathcal{F}_E$ satisfying (for instance) condition ii) of Lemma 2.20. This follows from (17). □

Combining above propositions we do not only characterize the freeness and topological freeness of $(\widehat{\mathcal{F}_E}, \widehat{\mathcal{V}})$ but also spot out an interesting dichotomy concerning its core subsystem $(\widehat{G}_\infty, \widehat{\mathcal{V}})$, cf. Remark 3.18 below.

Theorem 3.17. *We have the following dynamical dichotomy:*

a) *either every nonempty open set in $\widehat{G}_\infty$ contains uncountable number of nonperiodic points for $\widehat{\mathcal{V}}$; this holds if every loop in E has an exit, or*

b) *there are* $\widehat{\mathcal{V}}$*-periodic orbits* $O = \{\pi_{\mu^\infty}, \pi_{\sigma_E(\mu^\infty)}..., \pi_{\sigma_E^{n-1}(\mu^\infty)}\}$ *in* $\widehat{G}_\infty$ *forming open sets in* $\widehat{\mathcal{F}_E}$*; they correspond to loops without exits* μ.

In particular,

I) $\widehat{\mathcal{V}}$ *is topologically free if and only if every loop in* E *has an exit (condition (L)),*

II) $\widehat{\mathcal{V}}$ *is free if and only if every loop has an exit connected to this loop (the so called condition (K) introduced in* [21]*, see also* [4]*).*

Proof. Only item II) requires a comment. By Proposition 3.16 every closed $\widehat{\mathcal{V}}$-invariant set is of the form $\widehat{\mathcal{F}_E} \setminus \widehat{\mathcal{F}_V} = \widehat{\mathcal{F}_{E\setminus V}}$ for a hereditary and saturated subset $V \subset E^0$. Hence $\widehat{\mathcal{V}}$ is free if and only if every loop outside a hereditary saturated set V has exit outside V. One sees, cf. [4, p.318], that the latter is equivalent to the condition (K). □

Remark 3.18. If E has no sinks, then $\mathcal{F}_E = G_\infty$, and as we consider only finite graphs [20, Thm. 3.9] and Theorem 3.17 imply that $C^*(E)$ is purely infinite if and only if every nonempty open set in $\widehat{\mathcal{F}}_E$ contains uncountable number of nonperiodic points for $\widehat{\mathcal{V}}$. In particular, every $\widehat{\mathcal{V}}$-periodic orbit $O = \{\pi_{\mu^\infty}, \pi_{\sigma_E(\mu^\infty)}..., \pi_{\sigma_E^{n-1}(\mu^\infty)}\}$ yields a gauge invariant ideal J_O in $C^*(E)$ (generated by $\bigcap_{\pi\in\widehat{\mathcal{F}_E}\setminus O} \ker \pi$) which is not purely infinite. Indeed, if $v = s(\mu)$ is the source of the loop without exit μ, then $p_v C^*(E) p_v = p_v J_O p_v = C^*(s_\mu) \cong C(\mathbb{T})$ because s_μ is a unitary in $C^*(s_\mu)$ with the full spectrum, cf [20, Proof of Thm. 2.4].

To conclude let $B_k = \{a \in C^*(E) : \gamma_z(a) = z^k a \text{ for all } z \in \mathbb{T}\}$ be the k-th spectral subspace for the gauge action (10). Then $B_0 = \mathcal{F}_E$, $B_1 = \mathcal{F}_E s \mathcal{F}_E$ and $C^*(E) \cong B_0 \rtimes_{B_1} \mathbb{Z}$ via the gauge-invariant isomorphism, see Proposition 2.14. By Proposition 2.18 the inverse of the partial homeomorphism B_1-Ind of $\widehat{B}_0 = \widehat{\mathcal{F}}_E$ coincides with $\widehat{\mathcal{V}}$, and hence the results of the present section give the complete answer to the problem discussed in the introduction. In particular, Theorem 3.17 implies that condition (L) is equivalent to topological freeness of B_1-Ind. In other words, the uniqueness theorem of [17] when applied to graph C^*-algebras is equivalent to the Cuntz-Krieger uniqueness theorem.

References

[1] B. Abadie, S. Eilers, R. Exel. Morita equivalence for crossed products by Hilbert C^*-bimodules. *Trans. Amer. Math. Soc.* **350**(8) (1998), 3043-3054.

[2] A. B. Antonevich, V. I. Bakhtin, A. V. Lebedev. Crossed product of C^*-algebra by an endomorphism, coefficient algebras and transfer operators. *Math. Sbor.* **202**(9) (2011), 1253–1283

[3] V. I. Bakhtin, A. V. Lebedev. When a C^*-algebra is a coefficient algebra for a given endomorphism. Preprint, arXiv: math.OA/0502414

[4] T. Bates, D. Pask, I. Raeburn, W. Szymański. The C^*-algebras of row-finite graphs. *New York J. Math.* **6** (2000), 307-324.

[5] O. Bratteli. Inductive limits of finite dimensional C^*-algebras. *Trans. Amer. Math. Soc.* **171** (1972), 195-234.

[6] L.G. Brown, J. Mingo, N. Shen. Quasi-multipliers and embeddings of Hilbert C^*-modules. *Canad. J. Math.,* **71** (1994), 1150–1174.

[7] N. Brownlowe, I. Raeburn, S. Vittadello. Exel's crossed product for non-unital C*-algebras. *Math. Proc. Camb. Phil. Soc.* **149** (2010), 423-444.

[8] J. Cuntz, W. Krieger. A class of C^*-algebras and topological Markov chains. *Invetiones Math.* **56** (1980), 256-268.

[9] R. Exel. A new look at the crossed-product of a C^*-algebra by an endomorphism. *Ergodic Theory Dynam. Systems* **23** (2003), 1733-1750.

[10] R. Exel. Interactions. *J. Funct. Analysis* **244** (2007), 26 - 62.

[11] R. Exel. A new look at the crossed-product of a C^*-algebra by a semigroup of endomorphisms. *Ergodic Theory Dynam. Systems* **28** (2008), 749-789.

[12] R. Exel. Interactions and dynamical systems of type (n, m) - a case study. Preprint, arXiv:1212.5963

[13] R. Exel, J. Renault. Semigroups of local homeomorphisms and interaction groups. *Ergodic Theory Dynam. Systems* **27** (2007), 1737-1771.

[14] P. R. Halmos and L. J. Wallen. Powers of partial isometries. *J. Math. and Mech.* **19** (1970), 657–663.

[15] A. an Huef, I. Raeburn. Stacey Crossed Products Associated to Exel Systems. *Integr. Equ. Oper. Theory* **72** (2012), 537-561

[16] Ja A. Jeong; Gi Hyun Park. Topological entropy for the canonical completely positive maps on graph C^*-algebras. *Bull. Austral. Math. Soc.* **70**(1) (2004), 101–116.

[17] B.K. Kwaśniewski. Topological freeness for Hilbert bimodules. Accepted in *Israel J. Math.* Preprint arXiv:1212.0361.

[18] B. K. Kwaśniewski. On transfer operators for C^*-dynamical systems. *Rocky J. Math.* **42** (3) (2012), 919-938.

[19] B. K. Kwaśniewski. Uniqueness property for C*-algebras given by relations with circular symmetry. *Geometric Methods in Physics, XXX Workshop. Trends in Mathematics*, Birkhauser Verlag, Basel, 2012, 303-312.

[20] A. Kumjian, D. Pask, I. Raeburn. Cuntz-Krieger algebras of directed graphs. *Pacific J. Math.* **184** (1998), 161–174.

[21] A. Kumjian, D. Pask, I. Raeburn, J. Renault. Graphs, groupoids and Cuntz-Krieger algebras. *J. Funct. Anal.* **144** (1997), 505–541.

[22] G. K. Pedersen. *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

[23] M. V. Pimsner. A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$. *Fields Institute Communications* **12** (1997), 189–212.

[24] I. Raeburn, D. P. Williams. *Morita equivalence and continuous-trace C^*-algebras.* Amer. Math. Soc., Providence, 1998.

E-mail address: `bartoszk@math.uwb.edu.pl`

Institute of Mathematics, University of Bialystok, ul. Akademicka 2, PL-15-267 Bialystok, Poland